Lecture 8

Eigenvalues and eigenvectors

Definition 13.1: Let A be an $n \times n$ matrix, and let λ be a real number. A λ-*eigenvector* for A is a **nonzero** n-vector v with the property that   .
We say that λ is an *eigenvalue* of A if   .

- This is for   matrices only.
- If v is a λ-eigenvector, then Av points in the same direction as v (if $\lambda > 0$) or the opposite direction (if $\lambda < 0$) or $Av = 0$ (if $\lambda = 0$).
- Some things would work better if we considered complex eigenvalues, and eigenvectors in $\mathbb{C}^n$, even if the entries in A are real. However, we will stick with the real case for the moment.
- The equation $Av = \lambda v$ is equivalent to the homogeneous equation $(A - \lambda I_n)v = 0$. We can solve this by row-reducing $A - \lambda I_n$ to get a matrix B say. If B has a pivot in every column then (because it is square) it must be the identity, so the reduced equation $Bv = 0$ says $v = 0$, so there are no λ-eigenvectors. If B does not have a pivot in every column then there will be at least one independent variable, so the equation $Bv = 0$ will have some nonzero solutions, which are the λ-eigenvectors for A.

Eigenvector example

Consider the case

$$A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \qquad a = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \qquad b = \begin{bmatrix} 1 \\ -1 \end{bmatrix}.$$

We have

$$Aa = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \end{bmatrix} = 2a \qquad Ab = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} = 0b$$

so a is a 2-eigenvector and b is a 0-eigenvector, so 2 and 0 are eigenvalues. We claim that these are the only eigenvalues, or equivalently that when $\lambda \not\in \{0, 2\}$ the only solution to $(A - \lambda I_2)v = 0$ is $v = 0$, or equivalently that the matrix $A - \lambda I_2 = \begin{bmatrix} 1-\lambda & 1 \\ 1 & 1-\lambda \end{bmatrix}$ row-reduces to I_2.

Subtract $1 - \lambda$ times row 2 from row 1 to get $\begin{bmatrix} 0 & 1-(1-\lambda)^2 \\ 1 & 1-\lambda \end{bmatrix}$.

Here $1 - (1-\lambda)^2 = 2\lambda - \lambda^2 = \lambda(2-\lambda)$, which is nonzero because $\lambda \not\in \{0, 2\}$.

Divide the row 1 by this to get $\begin{bmatrix} 0 & 1 \\ 1 & 1-\lambda \end{bmatrix}$; more steps then give $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I_2$.

Eigenvector example

Consider

$$A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 2 & 2 & 2 \\ 0 & 0 & 3 & 3 \\ 0 & 0 & 0 & 4 \end{bmatrix} \qquad a = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \qquad b = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} \qquad c = \begin{bmatrix} 3 \\ 4 \\ 2 \\ 0 \end{bmatrix} \qquad d = \begin{bmatrix} 8 \\ 12 \\ 9 \\ 3 \end{bmatrix}.$$

We have

$$Ad = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 2 & 2 & 2 \\ 0 & 0 & 3 & 3 \\ 0 & 0 & 0 & 4 \end{bmatrix}\begin{bmatrix} 8 \\ 12 \\ 9 \\ 3 \end{bmatrix} = \begin{bmatrix} 32 \\ 48 \\ 36 \\ 12 \end{bmatrix} = 4d,$$

which means that d is a 4-eigenvector for A, and 4 is an eigenvalue of A. Equally direct calculation shows that $Aa = a$ and $Ab = 2b$ and $Ac = 3c$, so a, b and c are also eigenvectors, and ,   and   are also eigenvalues of A. Using the general theory that we will discuss below, we can show that

(a) The only -eigenvectors are the nonzero multiples of a.
(b) The only -eigenvectors are the nonzero multiples of b.
(c) The only -eigenvectors are the nonzero multiples of c.
(d) The only 4-eigenvectors are the nonzero multiples of d.
(e) There are no more eigenvalues: if λ is a real number other than , ,   and , then the equation $Av = \lambda v$ has   as the only solution, so there are no λ-eigenvectors.

The characteristic polynomial

Definition 13.8: Let A be an $n \times n$ matrix. We define $\chi_A(t) = \det(A - t\,I_n)$ (where I_n is the identity matrix). This is the *characteristic polynomial* of A.

Example 13.9: For $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ we have $A - tI_2 = \begin{bmatrix} \quad & \quad \\ \quad & \quad \end{bmatrix}$ so

$$\chi_A(t) = \det \begin{bmatrix} \quad & \quad \\ \quad & \quad \end{bmatrix} = (a-t)(d-t) - bc = t^2 - (a+d)t + \quad\quad .$$

When $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ we have

$$\chi_A(t) = t^2 - (1+4)t + (1 \times 4 - 2 \times 3) = t^2 - 5t - 2.$$

Theorem 13.11: The eigenvalues of A are the roots of the characteristic polynomial.

Characteristic polynomial example

Consider $A = \begin{bmatrix} 2 & -1 & 2 \\ -1 & 3 & -1 \\ 2 & -1 & 2 \end{bmatrix}$, so $\chi_A(t) = \det \begin{bmatrix} 2-t & -1 & 2 \\ -1 & 3-t & -1 \\ 2 & -1 & 2-t \end{bmatrix}$

$$= (2-t)\det \begin{bmatrix} 3-t & -1 \\ -1 & 2-t \end{bmatrix} - (-1)\det \begin{bmatrix} -1 & -1 \\ 2 & 2-t \end{bmatrix} + 2\det \begin{bmatrix} -1 & 3-t \\ 2 & -1 \end{bmatrix}$$

$$\det \begin{bmatrix} 3-t & -1 \\ -1 & 2-t \end{bmatrix} = (3-t)(2-t) - (-1)(-1) = t^2 - 5t + 5$$

$$\det \begin{bmatrix} -1 & -1 \\ 2 & 2-t \end{bmatrix} = (-1)(2-t) - (-1)(2) = t$$

$$\det \begin{bmatrix} -1 & 3-t \\ 2 & -1 \end{bmatrix} = (-1)(-1) - (3-t)(2) = 2t - 5$$

$$\chi_A(t) = (2-t)(t^2 - 5t + 5) + t + 2(2t-5) = -t^3 + 7t^2 - 10t$$
$$= -t(t-2)(t-5).$$

The eigenvalues of A are the roots of $\chi_A(t)$, namely $\quad$, $\quad$ and $\quad$.

Eigenvalue example

Consider $A = \begin{bmatrix} -1 & 1 & 0 \\ -1 & 0 & 1 \\ -1 & 0 & 0 \end{bmatrix}$, so $\chi_A(t) = \det \begin{bmatrix} -1-t & 1 & 0 \\ -1 & -t & 1 \\ -1 & 0 & -t \end{bmatrix}$

$$= (-1-t)\det \begin{bmatrix} -t & 1 \\ 0 & -t \end{bmatrix} - \det \begin{bmatrix} -1 & 1 \\ -1 & -t \end{bmatrix} + 0\det \begin{bmatrix} -1 & -t \\ -1 & 0 \end{bmatrix}$$

$$= -t^2(1+t) - (t+1) + 0 = -(1+t^2)(1+t).$$

Eigenvector example

$$A = \begin{bmatrix} -1 & 1 & 0 \\ -1 & 0 & 1 \\ -1 & 0 & 0 \end{bmatrix} \qquad \chi_A(t) = -(1+t^2)(1+t)$$

As $1 + t^2$ is always positive, the only way $-(1+t^2)(1+t)$ can be zero is if $t = -1$. Thus, the only real eigenvalue of A is -1. When $\lambda = -1$ we have

$$A - \lambda I_3 = A + I_3 = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 1 & 1 \\ -1 & 0 & 1 \end{bmatrix}.$$

To find an eigenvector of eigenvalue -1, solve $(A + I_3)u = 0$, or

$$\begin{bmatrix} 0 & 1 & 0 \\ -1 & 1 & 1 \\ -1 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix},$$

or ($y = 0$ and $-x + y + z = 0$ and $-x + z = 0$). These equations reduce to $x = z$ with $y = 0$, so $\begin{bmatrix} x & y & z \end{bmatrix} = z \begin{bmatrix} 1 & 0 & 1 \end{bmatrix}$. This means that the (-1)-eigenvectors are just the nonzero multiples of the vector $\begin{bmatrix} 1 & 0 & 1 \end{bmatrix}^T$.

General method for eigenvectors

Method 13.14: Suppose we have an $n \times n$ matrix A, and we want to find the eigenvalues and eigenvectors.

(a) Calculate the characteristic polynomial $\chi_A(t) = \det(A - tI_n)$.

(b) Find all the real roots of $\chi_A(t)$, and list them as $\lambda_1, \dots, \lambda_k$. These are the eigenvalues of A.

(c) For each eigenvalue λ_i, row reduce the matrix $A - \lambda_i I_n$ to get a matrix B.

(d) Read off solutions to the equation $Bu = 0$ (which is easy because B is in RREF). These are the λ_i-eigenvectors of the matrix A.

Eigenvector example

Consider the matrix

$$A = \begin{bmatrix} 16 & 2 & 1 & 1 \\ 2 & 16 & 1 & 1 \\ 1 & 1 & 16 & 2 \\ 1 & 1 & 2 & 16 \end{bmatrix}$$

We will take it as given here that $\chi_A(t) = (t-14)^2(t-16)(t-20)$. Thus, the eigenvalues of A are 14, 16 and 20. To find the eigenvectors of eigenvalue 14, we write down the matrix $A - 14I_4$ and row-reduce it to get a matrix B as follows:

$$\begin{bmatrix} 2 & 2 & 1 & 1 \\ 2 & 2 & 1 & 1 \\ 1 & 1 & 2 & 2 \\ 1 & 1 & 2 & 2 \end{bmatrix} \to \begin{bmatrix} 0 & 0 & -3 & -3 \\ 0 & 0 & -3 & -3 \\ 1 & 1 & 2 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix} \to \begin{bmatrix} 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

If we write $u = \begin{bmatrix} a & b & c & d \end{bmatrix}^T$, then the equation $Bu = 0$ just gives $a + b = c + d = 0$, so $a = -b$ and $c = -d$ (with b and d arbitrary), so

$$u = \begin{bmatrix} -b & b & -d & d \end{bmatrix}^T$$

for some $b, d \in \mathbb{R}$. The eigenvectors of eigenvalue 14 are precisely the nonzero vectors of the above form. (Recall that eigenvectors are nonzero, by definition.)

Nasty eigenvalues

Using Maple, we find that one eigenvalue of the matrix

$$A = \begin{bmatrix} -1 & 0 & 0 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & 1 & -1 & 0 \\ -1 & -1 & -1 & -1 \end{bmatrix} \qquad \text{is}$$

This level of complexity is quite normal, even for matrices whose entries are all 0 or ± 1. Most examples in this course are carefully constructed to have simple eigenvalues and eigenvectors, but you should be aware that this is not typical. The methods that we discuss will work perfectly well for all matrices, but in practice we need to use computers to do the calculations. Also, it is rarely useful to work with exact expressions for the eigenvalues when they are as complicated as those above. Instead we should use the numerical approximation $\lambda \simeq 1.496698205$.

Eigenvector example

Consider $A = \begin{bmatrix} 3 & 0 & 0 & 2 \\ 0 & 0 & 2 & 0 \\ 0 & 2 & 0 & 0 \\ 2 & 0 & 0 & 0 \end{bmatrix}$. We will take it as given that $\chi_A(t) = (t+1)(t+2)(t-2)(t-4)$, so the eigenvalues are .
To find the eigenvectors of eigenvalue 2, we write down the matrix $A - 2I_4$ and row-reduce it to get a matrix B in RREF:

$$\begin{bmatrix} 1 & 0 & 0 & 2 \\ 0 & -2 & 2 & 0 \\ 0 & 2 & -2 & 0 \\ 2 & 0 & 0 & -2 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 & 2 \\ 0 & -2 & 2 & 0 \\ 0 & 2 & -2 & 0 \\ 0 & 0 & 0 & -4 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 & 2 \\ 0 & 2 & -2 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -4 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} = B.$$

If we write $u = \begin{bmatrix} a & b & c & d \end{bmatrix}^T$, then the equation $Bu = 0$ just gives $a = b - c = d = 0$, so

$$u = \begin{bmatrix} \; & \; & \; & \; \end{bmatrix}^T = c \begin{bmatrix} \; & \; & \; & \; \end{bmatrix}^T.$$

for some $c \in \mathbb{R}$. The eigenvectors of eigenvalue 2 are precisely the nonzero vectors of the above form. In particular, the vector $\begin{bmatrix} \; & \; & \; & \; \end{bmatrix}^T$ is an eigenvector of eigenvalue 2.

Eigenvector example

We will find the eigenvalues and eigenvectors for $A = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 3 & 0 \\ 4 & 0 & 0 \end{bmatrix}$.

$$\chi_A(t) = \det \begin{bmatrix} -t & 0 & 1 \\ 0 & 3-t & 0 \\ 4 & 0 & -t \end{bmatrix} = -t \det \begin{bmatrix} 3-t & 0 \\ 0 & -t \end{bmatrix} + \det \begin{bmatrix} 0 & 3-t \\ 4 & 0 \end{bmatrix}$$
$$= -t^3 + 3t^2 + 4t - 12 = (4 - t^2)(t-3) = \qquad\qquad$$

Thus, the eigenvalues are $\quad$, $\quad$ and $\quad$.
For the eigenvectors $\begin{bmatrix} a & b & c \end{bmatrix}^T$ of eigenvalue -2:

$$A + 2I_3 = \begin{bmatrix} 2 & 0 & 1 \\ 0 & 5 & 0 \\ 4 & 0 & 2 \end{bmatrix} \to \begin{bmatrix} \; & \; & \; \\ \; & \; & \; \\ \; & \; & \; \end{bmatrix}$$

The eigenvectors of eigenvalue -2 are solutions to the equation

$$\begin{bmatrix} \; & \; & \; \\ \; & \; & \; \\ \; & \; & \; \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix},$$

or $a + c/2 = 0$ and $b = 0$. Take $c = 2$ to get the eigenvector $\begin{bmatrix} \; & \; & \; \end{bmatrix}^T$.

Eigenvector example

$$A = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 3 & 0 \\ 4 & 0 & 0 \end{bmatrix} \qquad \text{eigenvalues } -2,\ 2 \text{ and } 3$$

For the eigenvectors of eigenvalue 2:

$$A - 2I_3 = \begin{bmatrix} -2 & 0 & 1 \\ 0 & 1 & 0 \\ 4 & 0 & -2 \end{bmatrix} \to \begin{bmatrix} \; & \; & \; \\ \; & \; & \; \\ \; & \; & \; \end{bmatrix}$$

This gives $a - c/2 = 0$ and $b = 0$.
Take $c = 2$ to get the eigenvector $\begin{bmatrix} \; & \; & \; \end{bmatrix}^T$.

For the eigenvectors of eigenvalue 3:

$$A - 3I_3 = \begin{bmatrix} -3 & 0 & 1 \\ 0 & 0 & 0 \\ 4 & 0 & -3 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & -1/3 \\ 0 & 0 & 0 \\ 4 & 0 & -3 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & -1/3 \\ 0 & 0 & 0 \\ 0 & 0 & -5/3 \end{bmatrix} \to \begin{bmatrix} \; & \; & \; \\ \; & \; & \; \\ \; & \; & \; \end{bmatrix}$$

This gives $\qquad\qquad$. Take $b = 1$ to get the eigenvector $\begin{bmatrix} \; & \; & \; \end{bmatrix}^T$.

Eigenvector errors

Try to find an eigenvector u of eigenvalue 2 for the matrix $A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$:

$$A - 2I = \begin{bmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{bmatrix} \to \begin{bmatrix} -1 & 1 & 1 \\ 0 & 0 & 2 \\ 0 & 2 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Now $u = \begin{bmatrix} x & y & z \end{bmatrix}^T$ with $x = y = z = 0$, ie $u = 0$.
It is **not correct** to conclude that the zero vector is an eigenvector of eigenvalue 2 for A. **By definition**, the zero vector is not an eigenvector.
Instead, we conclude that either

(a) We made a mistake in the row-reduction; or
(b) We made a mistake in our determination of eigenvalues, and 2 is not an eigenvalue after all.

In an eigenvector problem, if you have found the eigenvalues correctly and row-reduced $A - \lambda I$ correctly then you will always end up with a matrix with a row of zeros (not the identity matrix).
If you make a mistake in finding the eigenvalues or in row-reducing $A - \lambda I$, then the most common outcome will be that you end up with the identity matrix.

Independence of eigenvectors

Proposition 13.19: Let A be a $d \times d$ matrix, and let $v_1, \dots, v_n$ be eigenvectors of A. Suppose that the corresponding eigenvalues $\lambda_1, \dots, \lambda_n$ are all different.
Then the list $v_1, \dots, v_n$ is linearly independent.

Proof for $n = 2$.
Suppose we have a linear relation $\qquad \alpha_1 v_1 + \alpha_2 v_2 = 0. \qquad$ (P)
We now multiply both sides of this vector equation by the matrix $A - \lambda_2 I$.

$$\alpha_1(\lambda_1 - \lambda_2)v_1 + \alpha_2(\lambda_2 - \lambda_2)v_2 = 0$$

As the number $\lambda_1 - \lambda_2$ and the vector v_1 are nonzero, we can conclude that $\alpha_1 = 0$. If we instead multiply equation (P) by $A - \lambda_1 I$ we get

$$\alpha_2(\lambda_2 - \lambda_1)v_2 = 0.$$

As the number $\lambda_2 - \lambda_1$ and the vector v_2 are nonzero, we can conclude that $\alpha_2 = 0$. We have now seen that $\alpha_1 = \alpha_2 = 0$, so the relation (P) is the trivial relation. As this works for any linear relation between v_1 and v_2, we see that these vectors are linearly independent. $\square$

Independence of eigenvectors

Proposition 13.19: Let A be a $d \times d$ matrix, and let $v_1, \dots, v_n$ be eigenvectors of A. Suppose that the corresponding eigenvalues $\lambda_1, \dots, \lambda_n$ are all different. Then the list $v_1, \dots, v_n$ is linearly independent.

Proof for $n = 3$.

Suppose we have a linear relation $\alpha_1 v_1 + \alpha_2 v_2 + \alpha_3 v_3 = 0.$ (P)

Multiply both sides by $A - \lambda_3 I$, then by $A - \lambda_2 I$

$$\alpha_1(\lambda_1 - \lambda_3)v_1 + \alpha_2(\lambda_2 - \lambda_3)v_2 + \alpha_3(\lambda_3 - \lambda_3)v_3 = 0$$

$$\alpha_1(\lambda_1 - \lambda_3)(\lambda_1 - \lambda_2)v_1 + \alpha_2(\lambda_2 - \lambda_3)(\lambda_2 - \lambda_2)v_2 + \alpha_3(\lambda_3 - \lambda_3)(\lambda_2 - \lambda_2)v_3 = 0$$

As the eigenvalues are all different $(\lambda_1 - \lambda_3)(\lambda_1 - \lambda_2) \neq 0$. As v_1 is an eigenvector it is nonzero. It follows that $\alpha_1 = 0$. Similarly, multiplying (P) by $(A - \lambda_1 I)(A - \lambda_3 I)$ makes the first and third terms go away leaving $\alpha_2(\lambda_2 - \lambda_1)(\lambda_2 - \lambda_3)v_2 = 0$ and so $\alpha_2 = 0$. Similarly, multiplying (P) by $(A - \lambda_1 I)(A - \lambda_2 I)$ gives $\alpha_3(\lambda_3 - \lambda_1)(\lambda_3 - \lambda_2)v_3 = 0$ and $\alpha_3 = 0$.
We now see that $\alpha_1 = \alpha_2 = \alpha_3 = 0$, so relation (P) is the trivial relation. This means that the list v_1, v_2, v_3 is linearly independent. □

A generalisation

Suppose we have:

- A $d \times d$ matrix A
- A list $\lambda_1, \dots, \lambda_r$ of distinct eigenvalues
- A linearly independent list $\mathcal{V}_1 = (v_{1,1}, \dots, v_{1,h_1})$ of eigenvectors, all with eigenvalue λ_1
- A linearly independent list $\mathcal{V}_2 = (v_{2,1}, \dots, v_{2,h_2})$ of eigenvectors, all with eigenvalue λ_2
- ....
- A linearly independent list $\mathcal{V}_r = (v_{r,1}, \dots, v_{r,h_r})$ of eigenvectors, all with eigenvalue λ_r

We can then combine the lists $\mathcal{V}_1, \dots, \mathcal{V}_r$ into a single list

$$\mathcal{W} = (v_{1,1}, \cdots, v_{1,h_1}, v_{2,1}, \cdots, v_{2,h_2}, \cdots, v_{r,1}, \cdots, v_{r,h_r}).$$

One can show that the combined list $\mathcal{W}$ is linearly independent.
The problem sheet asks you to prove this.

Independence of eigenvectors

Proposition 13.19: Let A be a $d \times d$ matrix, and let $v_1, \dots, v_n$ be eigenvectors of A. Suppose that the corresponding eigenvalues $\lambda_1, \dots, \lambda_n$ are all different. Then the list $v_1, \dots, v_n$ is linearly independent.

Proof for general n.

Suppose we have a linear relation $\alpha_1 v_1 + \cdots + \alpha_n v_n = 0.$ (P)

For any k, we can multiply (P) by the product of all the matrices $A - \lambda_i I$ for $i \neq k$. This makes all the terms go away except for the k'th term. All that is left is

$$\alpha_k \left(\prod_{i \neq k} (\lambda_k - \lambda_i) \right) v_k = 0.$$

As all the eigenvalues are assumed to be different, the product in brackets is nonzero, so we can divide to get $\alpha_k v_k = 0$. As $v_k \neq 0$ this gives $\alpha_k = 0$. This holds for all k, so relation (P) is the trivial relation. This means that the list $v_1, \dots, v_n$ is linearly independent. □