Lecture 10

Powers and eigenvectors

Let A be an $n \times n$ matrix. We can form the powers $A^2 = AA$, $A^3 = AAA$ and so on, and these are again $n \times n$ matrices. It is conventional to take $A^0 = I_n$ and $A^1 = A$.

Now let u be an eigenvector of eigenvalue λ.

$$A^0 u = I_n u = u$$
$$A^1 u = Au = \lambda u$$
$$A^2 u = A.Au = A.\lambda u = \lambda A u = \lambda^2 u$$
$$A^3 u = A. \qquad = \qquad = \qquad = $$
$$A^4 u = A. \qquad = \qquad = \qquad = $$

and in general $A^k u = \lambda^k u$ for all $k \geq 0$.

This is a key point in many applications of eigenvalues and eigenvectors.

Powers of diagonalised matrices

Proposition 14.9: Suppose we have a diagonalisation $A = UDU^{-1}$, where $D = \text{diag}(\lambda_1, \dots, \lambda_n)$ say. Then for all $k \geq 0$ we have $D^k = \text{diag}(\lambda_1^k, \dots, \lambda_n^k)$ and

$$A^k = UD^kU^{-1} = U \; \text{diag}(\lambda_1^k, \dots, \lambda_n^k) \; U^{-1}.$$

Proof: For example:

$$A^4 = (UDU^{-1})^4 = (UDU^{-1})(UDU^{-1})(UDU^{-1})(UDU^{-1})$$
$$= UD(U^{-1}U)D(U^{-1}U)D(U^{-1}U)DU^{-1} = UDDDDU^{-1} = UD^4U^{-1}$$

It is clear that the general case works the same way, so $A^k = UD^kU^{-1}$ for all k. (More formal proof by induction.) Next:

$$\text{diag}(\lambda_1, \dots, \lambda_n)\,\text{diag}(\mu_1, \dots, \mu_n) = \text{diag}(\lambda_1\mu_1, \dots, \lambda_n\mu_n).$$

It follows that

$$D^k = \text{diag}(\lambda_1, \dots, \lambda_n)^k = \text{diag}(\lambda_1^k, \dots, \lambda_n^k).$$

(Again, a formal proof would go by induction on k.)

Diagonalisation example

We will diagonalise the matrix $A = \begin{bmatrix} 3 & 2 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & -2 \\ 0 & 0 & 0 & -3 \end{bmatrix}$ and thus find A^k.

As $A - tI_4$ is upper-triangular we see that the determinant is ______. This gives

$$\chi_A(t) = \det(A - tI_4) = t^2(t-3)(t+3),$$

and it follows that the eigenvalues are $\lambda_1 = \lambda_2 =$ ___ and $\lambda_3 =$ ___ and $\lambda_4 =$ ___. Consider the vectors

$$u_1 = \begin{bmatrix} 1 \\ 0 \\ -3 \\ 0 \end{bmatrix} \qquad u_2 = \begin{bmatrix} 2 \\ -3 \\ 0 \\ 0 \end{bmatrix} \qquad u_3 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \qquad u_4 = \begin{bmatrix} 2 \\ -3 \\ -6 \\ -9 \end{bmatrix}$$

It is straightforward to check that $Au_1 = Au_2 = 0$ and $Au_3 = 3u_3$ and $Au_4 = -3u_4$, so the vectors u_i are eigenvectors for A, with eigenvalues 0, 0, 3 and -3 respectively.
(These vectors were found by row-reducing the matrices $A - \lambda_i I_4$.)

Diagonalisation example

$\lambda_1 = \lambda_2 = 0$ and $\lambda_3 = 3$ and $\lambda_4 = -3$; $Au_i = \lambda_i u_i$ where

$$u_1 = \begin{bmatrix} 1 \\ 0 \\ -3 \\ 0 \end{bmatrix} \qquad u_2 = \begin{bmatrix} 2 \\ -3 \\ 0 \\ 0 \end{bmatrix} \qquad u_3 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \qquad u_4 = \begin{bmatrix} 2 \\ -3 \\ -6 \\ -9 \end{bmatrix}$$

Now put

$$U = [u_1|u_2|u_3|u_4] = \begin{bmatrix} \; \end{bmatrix} \qquad V = \frac{1}{9}\begin{bmatrix} 0 & 0 & -3 & 2 \\ 0 & -3 & 0 & 1 \\ 9 & 6 & 3 & -2 \\ 0 & 0 & 0 & -1 \end{bmatrix}$$

One can check that $UV = I_4$, so $U^{-1} = V$.
(V was found by row-reduction $[U|I_4] \to [I_4|V]$.)
We now have $A = UDU^{-1} = UDV$, where

$$D = \mathrm{diag}(\lambda_1, \lambda_2, \lambda_3, \lambda_4) = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & -3 \end{bmatrix}.$$

Diagonalisation example

$$A = UDU^{-1} = \frac{1}{9}\begin{bmatrix} 1 & 2 & 1 & 2 \\ 0 & -3 & 0 & -3 \\ -3 & 0 & 0 & -6 \\ 0 & 0 & 0 & -9 \end{bmatrix}\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & -3 \end{bmatrix}\begin{bmatrix} 0 & 0 & -3 & 2 \\ 0 & -3 & 0 & 1 \\ 9 & 6 & 3 & -2 \\ 0 & 0 & 0 & -1 \end{bmatrix}$$

$$A^k = UD^kU^{-1} = \frac{1}{9}\begin{bmatrix} 1 & 2 & 1 & 2 \\ 0 & -3 & 0 & -3 \\ -3 & 0 & 0 & -6 \\ 0 & 0 & 0 & -9 \end{bmatrix}\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & \; & 0 \\ 0 & 0 & 0 & \; \end{bmatrix}\begin{bmatrix} 0 & 0 & -3 & 2 \\ 0 & -3 & 0 & 1 \\ 9 & 6 & 3 & -2 \\ 0 & 0 & 0 & -1 \end{bmatrix}$$

$$= 3^{k-2}\begin{bmatrix} 1 & 2 & 1 & 2 \\ 0 & -3 & 0 & -3 \\ -3 & 0 & 0 & -6 \\ 0 & 0 & 0 & -9 \end{bmatrix}\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 9 & 6 & 3 & -2 \\ 0 & 0 & 0 & (-1)^{k+1} \end{bmatrix}$$

$$= 3^{k-2}\begin{bmatrix} 9 & 6 & 3 & -2(1+(-1)^k) \\ 0 & 0 & 0 & \; \\ 0 & 0 & 0 & \; \\ 0 & 0 & 0 & 9(-1)^k \end{bmatrix}$$

Diagonalisation example

We will diagonalise the matrix $A = \begin{bmatrix} 2 & 2 & 2 & 2 \\ 2 & 5 & 5 & 2 \\ 2 & 5 & 5 & 2 \\ 2 & 2 & 2 & 2 \end{bmatrix}$. Recall $\chi_A(t) = \det(B)$,

where

$$B = A - tI_4 = \begin{bmatrix} 2-t & 2 & 2 & 2 \\ 2 & 5-t & 5 & 2 \\ 2 & 5 & 5-t & 2 \\ 2 & 2 & 2 & 2-t \end{bmatrix}.$$

Method 12.9: row-reduce B and keep track of row operation factors.

$$\begin{bmatrix} 2-t & 2 & 2 & 2 \\ 2 & 5-t & 5 & 2 \\ 2 & 5 & 5-t & 2 \\ 2 & 2 & 2 & 2-t \end{bmatrix} \to \begin{bmatrix} 2-t & 2 & 2 & 2 \\ 2 & 5-t & 5 & 2 \\ 0 & t & -t & 0 \\ t & 0 & 0 & -t \end{bmatrix} \longrightarrow \begin{bmatrix} 2-t & 2 & 2 & 2 \\ 2 & 5-t & 5 & 2 \\ 0 & 1 & -1 & 0 \\ 1 & 0 & 0 & -1 \end{bmatrix} \to$$

$$\begin{bmatrix} 0 & 0 & 4 & 4-t \\ 0 & 0 & 10-t & 4 \\ 0 & 1 & -1 & 0 \\ 1 & 0 & 0 & -1 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 0 & 10-t & 4 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 4 & 4-t \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 10-t & 4 \\ 0 & 0 & 4 & 4-t \end{bmatrix}$$

- Subtract row 1 from row 4, and row 2 from row 3.
- Multiply rows 3 and 4 by $1/t$ (factor $1/t^2$)
- Subtract multiples of rows 3 and 4 from rows 1 and 2.
- Swap rows 1 and 4 (factor -1); Swap rows 2 and 3 (factor -1).

Diagonalisation example

$$B = A - tI_4 \to C = \begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 10-t & 4 \\ 0 & 0 & 4 & 4-t \end{bmatrix}; \text{ product of factors } \mu = 1/t^2$$

Expand down the columns to get

$$\det(C) = \det\begin{bmatrix} 10-t & 4 \\ 4 & 4-t \end{bmatrix} = (10-t)(4-t)-16 = t^2-14t+24 = (t-2)(t-12).$$

Thus $\chi_A(t) = \det(B) = \det(C)/\mu =$.
This means that the eigenvalues of A are .

Diagonalisation example

$$B = A - tI_4 \to C = \begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 10-t & 4 \\ 0 & 0 & 4 & 4-t \end{bmatrix} \text{; eigenvalues } 2, 12, 0.$$

To find an eigenvector of eigenvalue 2 we need to row-reduce the matrix $A - 2I_4$, which is just the matrix B with $t = 2$. We can therefore substitute $t = 2$ in C and then perform a few more steps to complete the row-reduction.

$$A - 2I_4 \to \begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 8 & 4 \\ 0 & 0 & 4 & 2 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & 1/2 \\ 0 & 0 & 0 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 1/2 \\ 0 & 0 & 1 & 1/2 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

The eigenvector $u_1 = \begin{bmatrix} w & x & y & z \end{bmatrix}^T$ of eigenvalue 2 must therefore satisfy $w - z = x + z/2 = y + z/2 = 0$, so $u_1 = z\begin{bmatrix} 1 & -1/2 & -1/2 & 1 \end{bmatrix}^T$, with z arbitrary. It will be convenient to take $z = 2$ so $u_1 = \begin{bmatrix} 2 & -1 & -1 & 2 \end{bmatrix}^T$.

Diagonalisation example

$$B = A - tI_4 \to C = \begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 10-t & 4 \\ 0 & 0 & 4 & 4-t \end{bmatrix} \text{; eigenvalues } 2, 12, 0.$$

To find an eigenvector of eigenvalue 12 we need to row-reduce the matrix $A - 12I_4$, which is just the matrix B with $t = 12$. We can therefore substitute $t = 12$ in C and then perform a few more steps to complete the row-reduction.

$$A - 12I_4 \to \begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & -2 & 4 \\ 0 & 0 & 4 & -8 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

The eigenvector $u_2 = \begin{bmatrix} w & x & y & z \end{bmatrix}^T$ of eigenvalue 12 must therefore satisfy $w - z = x - 2z = y - 2z = 0$, so $u_2 = z\begin{bmatrix} 1 & 2 & 2 & 1 \end{bmatrix}^T$, with z arbitrary. It will be convenient to take $z = 1$ so $u_2 = \begin{bmatrix} 1 & 2 & 2 & 1 \end{bmatrix}^T$.

Diagonalisation example

$$B = A - tI_4 \to C = \begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 10-t & 4 \\ 0 & 0 & 4 & 4-t \end{bmatrix} \text{; eigenvalues } 2, 12, 0.$$

Finally, we need to find the eigenvectors of eigenvalue 0. Our reduction $B \to C$ involved division by t, so it is not valid in this case where $t = 0$. We must therefore start again and row-reduce the matrix $A - 0I_4 = A$ directly, but that is easy:

$$\begin{bmatrix} 2 & 2 & 2 & 2 \\ 2 & 5 & 5 & 2 \\ 2 & 5 & 5 & 2 \\ 2 & 2 & 2 & 2 \end{bmatrix} \to \begin{bmatrix} 2 & 2 & 2 & 2 \\ 0 & 3 & 3 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

We conclude that the eigenvectors of eigenvalue 0 are the vectors $\begin{bmatrix} w & x & y & z \end{bmatrix}^T$ with $w + z = x + y = 0$. These form a two-dimensional space, and the vectors

$$u_3 = \begin{bmatrix} 1 & 0 & 0 & -1 \end{bmatrix}^T \qquad u_4 = \begin{bmatrix} 0 & 1 & -1 & 0 \end{bmatrix}^T$$

form a basis.

Diagonalisation example

$$\begin{aligned} \lambda_1 &= 2 \\ \lambda_2 &= 12 \\ \lambda_3 &= 0 \\ \lambda_4 &= 0 \end{aligned} \qquad u_1 = \begin{bmatrix} 2 \\ -1 \\ -1 \\ 2 \end{bmatrix} \qquad u_2 = \begin{bmatrix} 1 \\ 2 \\ 2 \\ 1 \end{bmatrix} \qquad u_3 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ -1 \end{bmatrix} \qquad u_4 = \begin{bmatrix} 0 \\ 1 \\ -1 \\ 0 \end{bmatrix}$$

Now put

$$U = [u_1|u_2|u_3|u_4] = \begin{bmatrix} 2 & 1 & 1 & 0 \\ -1 & 2 & 0 & 1 \\ -1 & 2 & 0 & -1 \\ 2 & 1 & -1 & 0 \end{bmatrix} \qquad D = \begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & 12 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Row reduce $[U|I_4] \to [I_4|U^{-1}]$. Answer is

$$U^{-1} = \frac{1}{10}\begin{bmatrix} 2 & -1 & -1 & 2 \\ 1 & 2 & 2 & 1 \\ 5 & 0 & 0 & -5 \\ 0 & 5 & -5 & 0 \end{bmatrix}.$$

We now have a diagonalisation $A = UDU^{-1}$.

Diagonalisation example

$$\begin{bmatrix}2&2&2&2\\2&5&5&2\\2&5&5&2\\2&2&2&2\end{bmatrix} = \frac{1}{10}\begin{bmatrix}2&1&1&0\\-1&2&0&1\\-1&2&0&-1\\2&1&-1&0\end{bmatrix}\begin{bmatrix}2&0&0&0\\0&12&0&0\\0&0&0&0\\0&0&0&0\end{bmatrix}\begin{bmatrix}2&-1&-1&2\\1&2&2&1\\5&0&0&-5\\0&5&-5&0\end{bmatrix}$$

$$\begin{bmatrix}2&2&2&2\\2&5&5&2\\2&5&5&2\\2&2&2&2\end{bmatrix}^n = \frac{1}{10}\begin{bmatrix}2&1&1&0\\-1&2&0&1\\-1&2&0&-1\\2&1&-1&0\end{bmatrix}\begin{bmatrix}2^n&0&0&0\\0&12^n&0&0\\0&0&0&0\\0&0&0&0\end{bmatrix}\begin{bmatrix}2&-1&-1&2\\1&2&2&1\\5&0&0&-5\\0&5&-5&0\end{bmatrix}$$

$$= \frac{1}{10}\begin{bmatrix}2&1&1&0\\-1&2&0&1\\-1&2&0&-1\\2&1&-1&0\end{bmatrix}\begin{bmatrix}2^{n+1}&-2^n&-2^n&2^{n+1}\\12^n&2\times12^n&2\times12^n&12^n\\0&0&0&0\\0&0&0&0\end{bmatrix}$$

$$= \frac{1}{10}\begin{bmatrix}&&2(12^n-2^n)&12^n+2^{n+2}\\2(12^n-2^n)&4\times12^n+2^n&4\times12^n+2^n&2(12^n-2^n)\\2(12^n-2^n)&4\times12^n+2^n&4\times12^n+2^n&2(12^n-2^n)\\12^n+2^{n+2}&2(12^n-2^n)&2(12^n-2^n)&12^n+2^{n+2}\end{bmatrix}.$$

Diagonalisation example

$$\begin{bmatrix}2&2&2&2\\2&5&5&2\\2&5&5&2\\2&2&2&2\end{bmatrix}^n = \frac{1}{10}\begin{bmatrix}12^n+2^{n+2}&2(12^n-2^n)&2(12^n-2^n)&12^n+2^{n+2}\\2(12^n-2^n)&4\times12^n+2^n&4\times12^n+2^n&2(12^n-2^n)\\2(12^n-2^n)&4\times12^n+2^n&4\times12^n+2^n&2(12^n-2^n)\\12^n+2^{n+2}&2(12^n-2^n)&2(12^n-2^n)&12^n+2^{n+2}\end{bmatrix}$$

$$\sim \frac{12^n}{10}\begin{bmatrix}1&2&2&1\\2&4&4&2\\2&4&4&2\\1&2&2&1\end{bmatrix}.$$

(because 12^n grows much faster than 2^n when n is large).