## Lecture 11

## Systems of differential equations

- If $\dot{x} = ax$ with $x = c$ at $t = 0$, then $x = c\,e^{at}$.
- If $\dot{x}_i = a_i x_i$ with $x_i = c_i$ at $t = 0$ (for $i = 1, 2, 3$), then $x_i = c_i\,e^{a_i t}$
- Put

$$x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \qquad c = \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} \qquad D = \begin{bmatrix} a_1 & 0 & 0 \\ 0 & a_2 & 0 \\ 0 & 0 & a_3 \end{bmatrix} \qquad e^{Dt} = \begin{bmatrix} e^{a_1 t} & 0 & 0 \\ 0 & e^{a_2 t} & 0 \\ 0 & 0 & e^{a_3 t} \end{bmatrix}$$

  The equations are $\dot{x} = Dx$ with $x = c$ at $t = 0$; the solution is $x = e^{Dt}c$.
- Suppose instead $x = c$ at $t = 0$ with

$$\begin{aligned} \dot{x}_1 &= a_{11}x_1 + a_{12}x_2 + a_{13}x_3 \\ \dot{x}_2 &= a_{21}x_1 + a_{22}x_2 + a_{23}x_3 \\ \dot{x}_3 &= a_{31}x_1 + a_{32}x_2 + a_{33}x_3 \end{aligned} \qquad \text{so } \dot{x} = Ax \text{ where } A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}.$$

- To solve this, diagonalise $A = UDU^{-1}$ with $D = \mathrm{diag}(\lambda_1, \lambda_2, \lambda_3)$ say, so $\dot{x} = UDU^{-1}x$. Put $y = U^{-1}x$ and $d = U^{-1}c$ so $\dot{y} = U^{-1}\dot{x} = DU^{-1}x = Dy$, with $y = d$ at $t = 0$. This gives $y = e^{Dt}d$, where

$$e^{Dt} = \mathrm{diag}(e^{\lambda_1 t}, e^{\lambda_2 t}, e^{\lambda_3 t});$$

  so $x = Uy = Ue^{Dt}d = Ue^{Dt}U^{-1}c$.

## Differential equations example

If $\dot{x} = Ax$, $x = c$ at $t = 0$, $A = UDU^{-1}$, then $x = Ue^{Dt}U^{-1}c$ where $D = \mathrm{diag}(\lambda_1, \dots, \lambda_n)$ and $e^{Dt} = \mathrm{diag}(e^{\lambda_1 t}, \dots, e^{\lambda_n t})$.

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**Example 15.1:** Suppose $\dot{x}_1 = x_1 + x_2 + x_3$; $\quad \dot{x}_2 = 2x_2 + 2x_3$; $\quad \dot{x}_3 = 3x_3$ with $x_1 = x_2 = 0$ and $x_3 = 1$ at $t = 0$. This can be written as $\dot{x} = Ax$, where $A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 2 & 2 \\ 0 & 0 & 3 \end{bmatrix}$. By Example 14.6: $A = UDU^{-1}$, where

$$U = \begin{bmatrix} 1 & 1 & 3/2 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix} \qquad D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix} \qquad U^{-1} = \begin{bmatrix} 1 & -1 & 1/2 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{bmatrix}.$$

So $x = Ue^{Dt}U^{-1}c$, where $c = $ initial value $= \begin{bmatrix} 0 & 0 & 1 \end{bmatrix}^T$. Thus

$$x = \begin{bmatrix} 1 & 1 & 3/2 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} e^t & 0 & 0 \\ 0 & e^{2t} & 0 \\ 0 & 0 & e^{3t} \end{bmatrix} \begin{bmatrix} 1 & -1 & 1/2 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} e^t & e^{2t} & \frac{3}{2}e^{3t} \\ 0 & e^{2t} & 2e^{3t} \\ 0 & 0 & e^{3t} \end{bmatrix} \begin{bmatrix} 1/2 \\ -2 \\ 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{2}e^t - 2e^{2t} + \frac{3}{2}e^{3t} \\ -2e^{2t} + 2e^{3t} \\ e^{3t} \end{bmatrix}.$$

## Differential equations example

Suppose $\dot{x} = \dot{y} = \dot{z} = x + y + z$ with $x = z = 0$ and $y = 1$ at $t = 0$.

Thus $\dot{v} = Av$, where $v = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ and $A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$; $v = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$ at $t = 0$

The characteristic polynomial is

$$\chi_A(t) = \det \begin{bmatrix} 1-t & 1 & 1 \\ 1 & 1-t & 1 \\ 1 & 1 & 1-t \end{bmatrix} = 3t^2 - t^3 = t^2(3 - t).$$

Eigenvalues are $\lambda_1 = 0$, $\lambda_2 = 0$ and $\lambda_3 = 3$. If we put

$$u_1 = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} \qquad u_2 = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} \qquad u_3 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

we find that $Au_1 = Au_2 = 0$ and $Au_3 = 3u_3$. Thus, the vectors $u_i$ form a basis for $\mathbb{R}^3$ consisting of eigenvectors for $A$. This means that we have a diagonalisation $A = UDU^{-1}$, where

$$U = \begin{bmatrix} 1 & 0 & 1 \\ -1 & 1 & 1 \\ 0 & -1 & 1 \end{bmatrix} \qquad D = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 3 \end{bmatrix}.$$

## Differential equations example

$\dot{v} = UDU^{-1}v$ and $v = c$ at $t = 0$ where

$$U = \begin{bmatrix} 1 & 0 & 1 \\ -1 & 1 & 1 \\ 0 & -1 & 1 \end{bmatrix} \qquad D = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 3 \end{bmatrix} \qquad c = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

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We can find $U^{-1}$ by the following row-reduction:

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ -1 & 1 & 1 & 0 & 1 & 0 \\ 0 & -1 & 1 & 0 & 0 & 1 \end{array}\right] \to \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 3 & 1 & 1 & 1 \\ 0 & -1 & 1 & 0 & 0 & 1 \end{array}\right] \to$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 2/3 & -1/3 & -1/3 \\ 0 & 0 & 1 & 1/3 & 1/3 & 1/3 \\ 0 & -1 & 0 & -1/3 & -1/3 & 2/3 \end{array}\right] \to \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 2/3 & -1/3 & -1/3 \\ 0 & 1 & 0 & 1/3 & 1/3 & -2/3 \\ 0 & 0 & 1 & 1/3 & 1/3 & 1/3 \end{array}\right].$$

The conclusion is that

$$U^{-1} = \frac{1}{3}\begin{bmatrix} 2 & -1 & -1 \\ 1 & 1 & -2 \\ 1 & 1 & 1 \end{bmatrix}.$$

## Differential equations example

$\dot{v} = UDU^{-1}v$ and $v = c$ at $t = 0$ where

$$U = \begin{bmatrix} 1 & 0 & 1 \\ -1 & 1 & 1 \\ 0 & -1 & 1 \end{bmatrix} \quad D = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 3 \end{bmatrix} \quad U^{-1} = \frac{1}{3}\begin{bmatrix} 2 & -1 & -1 \\ 1 & 1 & -2 \\ 1 & 1 & 1 \end{bmatrix} \quad c = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

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The solution to our differential equation is now $v = Ue^{Dt}U^{-1}c$:

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \frac{1}{3}\begin{bmatrix} 1 & 0 & 1 \\ -1 & 1 & 1 \\ 0 & -1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & e^{3t} \end{bmatrix}\begin{bmatrix} 2 & -1 & -1 \\ 1 & 1 & -2 \\ 1 & 1 & 1 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

$$= \frac{1}{3}\begin{bmatrix} 1 & 0 & e^{3t} \\ -1 & 1 & e^{3t} \\ 0 & -1 & e^{3t} \end{bmatrix}\begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} (e^{3t}-1)/3 \\ (e^{3t}+2)/3 \\ (e^{3t}-1)/3 \end{bmatrix}.$$

## Another approach to the same problem

- Again consider $\dot{x} = \dot{y} = \dot{z} = x + y + z$ with $x = z = 0$, $y = 1$ at $t = 0$.
- Put $a = x - y$. Subtracting $\dot{x} = x + y + z$ and $\dot{y} = x + y + z$ we get $\dot{a} = 0$, so $a$ is constant. At $t = 0$ we have $a = 0 - 1 = -1$, so $a = -1$ for all $t$.
- Put $b = y - z$. Subtracting $\dot{y} = x + y + z$ and $\dot{z} = x + y + z$ we get $\dot{b} = 0$, so $b$ is constant. At $t = 0$ we have $b = 1 - 0 = 1$, so $b = 1$ for all $t$.
- Put $c = x + y + z$. Adding the equations $\dot{x} = c$ and $\dot{y} = c$ and $\dot{z} = c$ gives $\dot{c} = 3c$, so $c = c_0 e^{3t}$ for some constant $c_0$. At $t = 0$ we have $c = 0 + 1 + 0 = 1$, so $c_0 = 1$, so $c = e^{3t}$.
- We now have $x - y = -1$ and $y - z = 1$ and $x + y + z = e^{3t}$. These can be solved algebraically to get

  $$x = \frac{e^{3t}-1}{3} \qquad y = \frac{e^{3t}+2}{3} \qquad z = \frac{e^{3t}-1}{3}$$

  as before.
- Here we found appropriate quantities $a$, $b$ and $c$ by inspection. The eigenvalue method effectively does the same thing in a more systematic way.

## Solving difference equations

**Problem:** find a formula for the sequence where $a_0 = -1$, $a_1 = 0$, and $a_{i+2} = 6a_{i+1} - 8a_i$ for all $i \geq 0$.

$$a_2 = 6a_1 - 8a_0 = 6 \times 0 - 8 \times (-1) = 8$$
$$a_3 = 6a_2 - 8a_1 = 6 \times 8 - 8 \times 0 = 48$$
$$a_4 = 6a_3 - 8a_2 = 6 \times 48 - 8 \times 8 = 224 \qquad \text{etc.}$$

Vector formulation: put $v_i = \begin{bmatrix} a_i \\ a_{i+1} \end{bmatrix} \in \mathbb{R}^2$, so $v_0 = \begin{bmatrix} -1 \\ 0 \end{bmatrix}$ and

$$v_{n+1} = \begin{bmatrix} a_{n+1} \\ a_{n+2} \end{bmatrix} = \begin{bmatrix} a_{n+1} \\ 6a_{n+1} - 8a_n \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -8 & 6 \end{bmatrix}\begin{bmatrix} a_n \\ a_{n+1} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -8 & 6 \end{bmatrix} v_n.$$

We write $A = \begin{bmatrix} 0 & 1 \\ -8 & 6 \end{bmatrix}$, so the above reads $v_{n+1} = Av_n$. Thus $v_1 = Av_0$, $v_2 = Av_1 = A^2v_0$, $v_3 = Av_2 = A^3v_0$, $v_n = A^nv_0$.

We can be more explicit by finding the eigenvalues and eigenvectors of $A$.

## Solving difference equations

$$v_n = \begin{bmatrix} a_n \\ a_{n+1} \end{bmatrix} = A^n v_0 \qquad A = \begin{bmatrix} 0 & 1 \\ -8 & 6 \end{bmatrix} \qquad v_0 = \begin{bmatrix} -1 \\ 0 \end{bmatrix}.$$

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The characteristic polynomial is

$$\chi_A(t) = \det \begin{bmatrix} -t & 1 \\ -8 & 6-t \end{bmatrix} = t^2 - 6t + 8 = (t-2)(t-4),$$

so the eigenvectors are 2 and 4. Using the row-reductions

$$A - 2I = \begin{bmatrix} -2 & 1 \\ -8 & 4 \end{bmatrix} \to \begin{bmatrix} 1 & -1/2 \\ 0 & 0 \end{bmatrix} \qquad A - 4I = \begin{bmatrix} -4 & 1 \\ -8 & 2 \end{bmatrix} \to \begin{bmatrix} 1 & -1/4 \\ 0 & 0 \end{bmatrix}$$

we see that $u_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ and $u_2 = \begin{bmatrix} 1 \\ 4 \end{bmatrix}$ are eigenvectors of eigenvalues 2 and 4 (forming a basis for $\mathbb{R}^2$). We now have a diagonalisation $A = UDU^{-1}$, where

$$U = [u_1|u_2] = \begin{bmatrix} 1 & 1 \\ 2 & 4 \end{bmatrix} \qquad D = \begin{bmatrix} 2 & 0 \\ 0 & 4 \end{bmatrix} \qquad U^{-1} = \begin{bmatrix} 2 & -1/2 \\ -1 & 1/2 \end{bmatrix}.$$

This gives $v_n = A^n v_0 = UD^nU^{-1}v_0$.

## Solving difference equations

$$v_n = \begin{bmatrix} a_n \\ a_{n+1} \end{bmatrix} = A^n v_0 = UD^nU^{-1}v_0$$

$$U = \begin{bmatrix} 1 & 1 \\ 2 & 4 \end{bmatrix} \qquad D = \begin{bmatrix} 2 & 0 \\ 0 & 4 \end{bmatrix} \qquad U^{-1} = \begin{bmatrix} 2 & -1/2 \\ -1 & 1/2 \end{bmatrix} \qquad v_0 = \begin{bmatrix} -1 \\ 0 \end{bmatrix}.$$

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$$\begin{bmatrix} a_n \\ a_{n+1} \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 2 & 4 \end{bmatrix} \begin{bmatrix} 2^n & 0 \\ 0 & 4^n \end{bmatrix} \begin{bmatrix} 2 & -1/2 \\ -1 & 1/2 \end{bmatrix} \begin{bmatrix} -1 \\ 0 \end{bmatrix}$$
$$= \begin{bmatrix} 2^n & 4^n \\ 2^{n+1} & 4^{n+1} \end{bmatrix} \begin{bmatrix} -2 \\ 1 \end{bmatrix} = \begin{bmatrix} 4^n - 2^{n+1} \\ 4^{n+1} - 2^{n+2} \end{bmatrix}$$

Thus $a_n = 4^n - 2^{n+1} = 2^{2n} - 2^{n+1}$. We will check that this formula does indeed give the required properties:

$$a_0 = 2^0 - 2^1 = 1 - 2 = -1$$
$$a_1 = 2^2 - 2^2 = 0$$
$$6a_{i+1} - 8a_i = 6(2^{2i+2} - 2^{i+2}) - 8(2^{2i} - 2^{i+1}) = 24 \times 2^{2i} - 24 \times 2^i - 8 \times 2^{2i} + 16 \times 2^i$$
$$= 16 \times 2^{2i} - 8 \times 2^i = 2^{2i+4} - 2^{i+3} = 2^{2(i+2)} - 2^{(i+2)+1} = a_{i+2}.$$

## Another difference equation

We will find the sequence satisfying $b_0 = 3$ and $b_1 = 6$ and $b_2 = 14$ and

$$b_{i+3} = 6b_i - 11b_{i+1} + 6b_{i+2}.$$

The vectors $v_i = \begin{bmatrix} b_i & b_{i+1} & b_{i+2} \end{bmatrix}^T$ satisfy $v_0 = \begin{bmatrix} 3 & 6 & 14 \end{bmatrix}^T$ and

$$v_{i+1} = \begin{bmatrix} b_{i+1} \\ b_{i+2} \\ b_{i+3} \end{bmatrix} = \begin{bmatrix} b_{i+1} \\ b_{i+2} \\ 6b_i - 11b_{i+1} + 6b_{i+2} \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 6 & -11 & 6 \end{bmatrix} \begin{bmatrix} b_i \\ b_{i+1} \\ b_{i+2} \end{bmatrix} = Bv_i.$$

It follows that $v_n = B^n v_0$ for all $n$, and $b_n$ is the top entry in the vector $v_n$. Now diagonalise $B$. The characteristic polynomial is

$$\chi_B(t) = \det \begin{bmatrix} -t & 1 & 0 \\ 0 & -t & 1 \\ 6 & -11 & 6-t \end{bmatrix} = -t \det \begin{bmatrix} -t & 1 \\ -11 & 6-t \end{bmatrix} - \det \begin{bmatrix} 0 & 1 \\ 6 & 6-t \end{bmatrix}$$
$$= -t(t^2 - 6t + 11) - (-6) = 6 - 11t + 6t^2 - t^3 = (1-t)(2-t)(3-t),$$

so the eigenvalues are 1, 2 and 3.

## Another difference equation

$$v_n = \begin{bmatrix} b_n \\ b_{n+1} \\ b_{n+2} \end{bmatrix} = B^n \begin{bmatrix} 3 \\ 6 \\ 14 \end{bmatrix} \qquad B = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 6 & -11 & 6 \end{bmatrix} \text{ has eigenvalues } 1, 2, 3.$$

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Now find the eigenvectors:

$$B - I = \begin{bmatrix} -1 & 1 & 0 \\ 0 & -1 & 1 \\ 6 & -11 & 5 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix} \qquad u_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$
$$B - 2I = \begin{bmatrix} -2 & 1 & 0 \\ 0 & -2 & 1 \\ 6 & -11 & 4 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & -1/4 \\ 0 & 1 & -1/2 \\ 0 & 0 & 0 \end{bmatrix} \qquad u_2 = \begin{bmatrix} 1 \\ 2 \\ 4 \end{bmatrix}$$
$$B - 3I = \begin{bmatrix} -3 & 1 & 0 \\ 0 & -3 & 1 \\ 6 & -11 & 3 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & -1/9 \\ 0 & 1 & -1/3 \\ 0 & 0 & 0 \end{bmatrix} \qquad u_3 = \begin{bmatrix} 1 \\ 3 \\ 9 \end{bmatrix}.$$

$$B = UDU^{-1} \quad \text{where} \quad U = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 4 & 9 \end{bmatrix} \qquad D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

## Another difference equation

$$v_n = UD^nU^{-1}v_0 \qquad U = \begin{bmatrix}1&1&1\\1&2&3\\1&4&9\end{bmatrix} \qquad D = \begin{bmatrix}1&0&0\\0&2&0\\0&0&3\end{bmatrix} \qquad v_0 = \begin{bmatrix}3\\6\\14\end{bmatrix}$$

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Find $U^{-1}$ by the adjugate method:

$$\begin{array}{lll} m_{11}=6 & m_{12}=6 & m_{13}=2\\ m_{21}=5 & m_{22}=8 & m_{23}=3\\ m_{31}=1 & m_{32}=2 & m_{33}=1. \end{array}$$

$$\operatorname{adj}(U) = \begin{bmatrix}+m_{11}&-m_{21}&+m_{31}\\-m_{12}&+m_{22}&-m_{32}\\+m_{13}&-m_{23}&+m_{33}\end{bmatrix} = \begin{bmatrix}6&-5&1\\-6&8&-2\\2&-3&1\end{bmatrix}.$$

$$\det(U) = 1m_{11} - 1m_{12} + 1m_{13} = 6-6+2 = 2.$$

$$U^{-1} = \frac{\operatorname{adj}(U)}{\det(U)} = \begin{bmatrix}3&-5/2&1/2\\-3&4&-1\\1&-3/2&1/2\end{bmatrix}.$$

## Another difference equation

$$v_n = UD^nU^{-1}v_0 = \begin{bmatrix}1&1&1\\1&2&3\\1&4&9\end{bmatrix}\begin{bmatrix}1&0&0\\0&2^n&0\\0&0&3^n\end{bmatrix}\begin{bmatrix}3&-5/2&1/2\\-3&4&-1\\1&-3/2&1/2\end{bmatrix}\begin{bmatrix}3\\6\\14\end{bmatrix}$$

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$$= \begin{bmatrix}1&2^n&3^n\\1&2^{n+1}&3^{n+1}\\1&2^{n+2}&3^{n+2}\end{bmatrix}\begin{bmatrix}1\\1\\1\end{bmatrix} = \begin{bmatrix}1+2^n+3^n\\1+2^{n+1}+3^{n+1}\\1+2^{n+2}+3^{n+2}\end{bmatrix}.$$

Thus $b_n = 1+2^n+3^n$. Check:

$$b_0 = 1+1+1 = 3 \qquad b_1 = 1+2+3 = 6 \qquad b_2 = 1+4+9 = 14$$

$$\begin{aligned} 6b_i - 11b_{i+1} + 6b_{i+2} &= 6(1+2^i+3^i) - 11(1+2^{i+1}+3^{i+1}) + 6(1+2^{i+2}+3^{i+2})\\ &= (6-11+6) + (6-22+24)2^i + (6-33+54)3^i\\ &= 1 + 8\times 2^i + 27\times 3^i\\ &= 1 + 2^{i+3} + 3^{i+3} = b_{i+3}. \end{aligned}$$

## A slightly different method

Find eigenvalues and eigenvectors as before:

$$\begin{array}{l}\lambda_1 = 1\\ \lambda_2 = 2\\ \lambda_3 = 3\end{array} \qquad u_1 = \begin{bmatrix}1\\1\\1\end{bmatrix} \qquad u_2 = \begin{bmatrix}1\\2\\4\end{bmatrix} \qquad u_3 = \begin{bmatrix}1\\3\\9\end{bmatrix}.$$

Recall $v_n = UD^nU^{-1}v_0$. We do not need all of $U^{-1}$, we only need $U^{-1}v_0$.
This amounts to expressing $v_0$ as a linear combination of the columns $u_i$.
For this we row-reduce $[u_1|u_2|u_3|v_0]$:

$$\left[\begin{array}{ccc|c}1&1&1&3\\1&2&3&6\\1&4&9&14\end{array}\right] \to \left[\begin{array}{ccc|c}1&1&1&3\\0&1&2&3\\0&3&8&11\end{array}\right] \to \cdots \to \left[\begin{array}{ccc|c}1&0&0&1\\0&1&0&1\\0&0&1&1\end{array}\right].$$

The required coefficients appear in the last column: $v_0 = u_1 + u_2 + u_3$.
This can also be seen by inspection.
As $u_k$ is an eigenvector of eigenvalue $k$, we have $B^nu_k = k^nu_k$, so

$$v_n = B^nv_0 = B^nu_1 + B^nu_2 + B^nu_3 = \begin{bmatrix}1\\1\\1\end{bmatrix} + 2^n\begin{bmatrix}1\\2\\4\end{bmatrix} + 3^n\begin{bmatrix}1\\3\\9\end{bmatrix} = \begin{bmatrix}1+2^n+3^n\\1+2^{n+1}+3^{n+1}\\1+2^{n+2}+3^{n+2}\end{bmatrix}.$$

Moreover, $b_n$ is the top entry in $v_n$, so we again conclude that

$$b_n = 1+2^n+3^n.$$

## Fibonacci numbers

The Fibonacci numbers are given by $F_0 = 0$ and $F_1 = 1$ and $F_{n+2} = F_n + F_{n+1}$.
The vectors $v_i = \begin{bmatrix}F_i\\F_{i+1}\end{bmatrix}$ therefore satisfy $v_0 = \begin{bmatrix}0\\1\end{bmatrix}$ and

$$v_{n+1} = \begin{bmatrix}F_{n+1}\\F_{n+2}\end{bmatrix} = \begin{bmatrix}F_{n+1}\\F_n + F_{n+1}\end{bmatrix} = Av_n, \qquad \text{where} \qquad A = \begin{bmatrix}0&1\\1&1\end{bmatrix}.$$

It follows that $v_n = A^nv_0$. We have $\chi_A(t) = t^2 - t - 1$, which has roots $\lambda_1 = (1+\sqrt{5})/2$ and $\lambda_2 = (1-\sqrt{5})/2$. To find an eigenvector of eigenvalue $\lambda_1$, we must solve

$$\begin{bmatrix}0&1\\1&1\end{bmatrix}\begin{bmatrix}x\\y\end{bmatrix} = \lambda_1\begin{bmatrix}x\\y\end{bmatrix} \qquad \text{or} \qquad \begin{aligned} y &= \lambda_1 x\\ x+y &= \lambda_1 y\end{aligned}$$

Substituting $y = \lambda_1 x$ in $x + y = \lambda_1 y$ gives $x + \lambda_1 x = \lambda_1^2 x$, or $(\lambda_1^2 - \lambda_1 - 1)x = 0$, which is automatic, because $\lambda_1$ is a root of $t^2 - t - 1 = 0$.
Take $x = 1$ to get an eigenvector $u_1 = \begin{bmatrix}1\\\lambda_1\end{bmatrix}$ of eigenvalue $\lambda_1$.

Similarly, we have an eigenvector $u_2 = \begin{bmatrix}1\\\lambda_2\end{bmatrix}$ of eigenvalue $\lambda_2$.

## Fibonacci numbers

$$v_n = A^n \begin{bmatrix} 0 \\ 1 \end{bmatrix} \qquad u_k = \begin{bmatrix} 1 \\ \lambda_k \end{bmatrix} \qquad Au_k = \lambda_k u_k \qquad \begin{array}{l} \lambda_1 = (1+\sqrt{5})/2 \\ \lambda_2 = (1-\sqrt{5})/2 \end{array}$$

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We now need to find $\alpha$ and $\beta$ such that $\alpha u_1 + \beta u_2 = v_0$, or equivalently

$$\alpha \begin{bmatrix} 1 \\ \lambda_1 \end{bmatrix} + \beta \begin{bmatrix} 1 \\ \lambda_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \qquad \text{or} \qquad \begin{aligned} \beta &= -\alpha \\ \alpha(\lambda_1 - \lambda_2) &= 1. \end{aligned}$$

Now $\lambda_1 - \lambda_2 = \sqrt{5}$ so $\alpha = 1/\sqrt{5}$ and $\beta = -1/\sqrt{5}$ and $v_0 = (u_1 - u_2)/\sqrt{5}$.

$$v_n = A^n v_0 = \frac{A^n u_1 - A^n u_2}{\sqrt{5}} = \frac{\lambda_1^n u_1 - \lambda_2^n u_2}{\sqrt{5}} = \frac{1}{\sqrt{5}} \begin{bmatrix} \lambda_1^n - \lambda_2^n \\ \lambda_1^{n+1} - \lambda_2^{n+1} \end{bmatrix}.$$

Moreover, $F_n$ is the top entry in $v_n$, so we obtain the formula

$$F_n = \frac{\lambda_1^n - \lambda_2^n}{\sqrt{5}} = \frac{(1+\sqrt{5})^n - (1-\sqrt{5})^n}{2^n\,\sqrt{5}}.$$

It is also useful to note here that $\lambda_1 \simeq 1.618033988$ and $\lambda_2 \simeq -0.6180339880$. As $|\lambda_1| > 1$ and $|\lambda_2| < 1$ we see that $|\lambda_1^n| \to \infty$ and $|\lambda_2^n| \to 0$ as $n \to \infty$. When $n$ is large we can neglect the $\lambda_2$ term and we have $F_n \simeq \lambda_1^n/\sqrt{5}$.