Lecture 14

Subspaces

In $\mathbb{R}^2$ and $\mathbb{R}^3$, lines and planes are important, especially through the origin. We now discuss analogous structures in $\mathbb{R}^n$, where n may be bigger than 3.

Definition 19.1: A subset $V \subseteq \mathbb{R}^n$ is a *subspace* if

(a) The zero vector is an element of V.

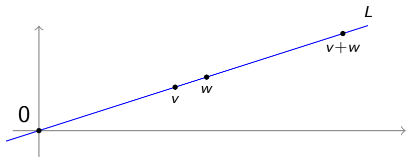
(b) Whenever v and w are two elements of V, the sum $v + w$ is also an element of V. (In other words, V is closed under addition.)

(c) Whenever v is an element of V and t is a real number, the vector tv is again an element of V. (In other words, V is closed under scalar multiplication.)

Subspace example

A subspace must contain 0, and be closed under addition and scalar multiplication.

Let L be the line in $\mathbb{R}^2$ with equation $y = x/\pi$.



- The point $0 = \begin{bmatrix} 0 & 0 \end{bmatrix}^T$ lies on L.
- Suppose we have $v, w \in L$, so $v = \begin{bmatrix} a & a/\pi \end{bmatrix}^T$ and $w = \begin{bmatrix} b & b/\pi \end{bmatrix}^T$ for some numbers a and b. Then $v + w = \begin{bmatrix} a+b & (a+b)/\pi \end{bmatrix}^T$, which again lies on L. Thus, L is closed under addition.
- Suppose again that $v \in L$, so $v = \begin{bmatrix} a & a/\pi \end{bmatrix}^T$ for some a. Suppose also that $t \in \mathbb{R}$. Then $tv = \begin{bmatrix} ta & ta/\pi \end{bmatrix}^T$, which again lies on L, so L is closed under scalar multiplication.

So L is a subspace.

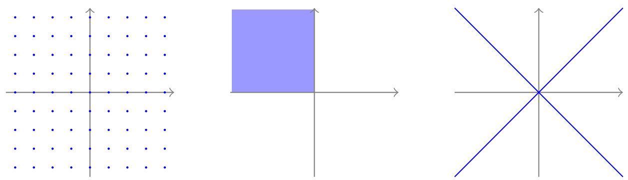
Subspace non-examples

Consider the following subsets of $\mathbb{R}^2$:

$$V_1 = \mathbb{Z}^2 = \left\{ \begin{bmatrix} x \\ y \end{bmatrix} \in \mathbb{R}^2 \mid x \text{ and } y \text{ are integers} \right\}$$

$$V_2 = \left\{ \begin{bmatrix} x \\ y \end{bmatrix} \in \mathbb{R}^2 \mid x \leq 0 \leq y \right\}$$

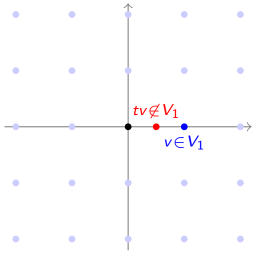
$$V_3 = \left\{ \begin{bmatrix} x \\ y \end{bmatrix} \in \mathbb{R}^2 \mid x^2 = y^2 \right\} = \left\{ \begin{bmatrix} x \\ y \end{bmatrix} \in \mathbb{R}^2 \mid x = \pm y \right\}.$$



V_1 $\quad$ V_2 $\quad$ V_3

None of these are subspaces.

V_1 is not a subspace

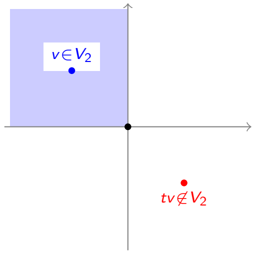


$$V_1 = \left\{ \begin{bmatrix} x \\ y \end{bmatrix} \in \mathbb{R}^2 \mid x \text{ and } y \text{ are integers} \right\}$$

It is clear that the zero vector has integer coordinates and so lies in V_1. Next, if v and w both have integer coordinates then so does $v+w$. In other words, if $v, w \in V_1$ then also $v+w \in V_1$, so V_1 is closed under addition. However, it is not closed under scalar multiplication. Indeed, if we take $v = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $t = 0.5$ then $v \in V_1$ and $t \in \mathbb{R}$ but the vector $tv = \begin{bmatrix} 0.5 \\ 0 \end{bmatrix}$ does not lie in V_1.

(This is generally the best way to prove that a set is not a subspace: provide a completely specific and explicit example where one of the conditions is not satisfied.)

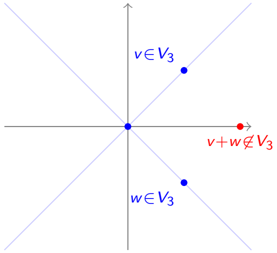
V_2 is not a subspace



$$V_2 = \left\{ \begin{bmatrix} x \\ y \end{bmatrix} \in \mathbb{R}^2 \mid x \leq 0 \leq y \right\}$$

As $0 \leq 0 \leq 0$ we see that $0 \in V_2$.
Suppose we have vectors $v = \begin{bmatrix} x & y \end{bmatrix}^T$ and $v' = \begin{bmatrix} x' & y' \end{bmatrix}^T$ in V_2,
so $x \leq 0 \leq y$ and $x' \leq 0 \leq y'$. As $x, x' \leq 0$ it follows that $x + x' \leq 0$.
As $y, y' \geq 0$ it follows that $y + y' \geq 0$. This means that the sum
$v + v' = \begin{bmatrix} x + x' & y + y' \end{bmatrix}^T$ is again in V_2, so V_2 is closed under addition.
However, it is not closed under scalar multiplication.
Indeed, if we take $v = \begin{bmatrix} -1 & 1 \end{bmatrix}^T$ and $t = -1$ then $v \in V_2$ and $t \in \mathbb{R}$ but the vector $tv = \begin{bmatrix} 1 & -1 \end{bmatrix}^T$ does not lie in V_2.

V_3 is not a subspace



$$V_3 = \left\{ \begin{bmatrix} x \\ y \end{bmatrix} \in \mathbb{R}^2 \mid x^2 = y^2 \right\} = \left\{ \begin{bmatrix} x \\ y \end{bmatrix} \in \mathbb{R}^2 \mid x = \pm y \right\}.$$

It is again clear that $0 \in V_3$.
Now suppose we have $v = \begin{bmatrix} x & y \end{bmatrix}^T \in V_3$ (so $x^2 = y^2$) and $t \in \mathbb{R}$.
It follows that $(tx)^2 = t^2x^2 = t^2y^2 = (ty)^2$,
so the vector $tv = \begin{bmatrix} tx & ty \end{bmatrix}^T$ again lies in V_3.
This means that V_3 is closed under scalar multiplication.
However, it is not closed under addition,
because the vectors $v = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $w = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ lie in V_3,
but $v + w$ does not.

Two extreme cases

(a) The set $\{0\}$ (just consisting of the zero vector) is a subspace of $\mathbb{R}^n$.
(b) The whole set $\mathbb{R}^n$ is a subspace of itself.

Linear combinations in subspaces

Proposition 19.6: Let V be a subspace of $\mathbb{R}^n$. Then any linear combination of elements of V is again in V.

Proof.
Suppose we have elements $v_1, \dots, v_k \in V$, and suppose that w is a linear combination of the v_i, say $w = \sum_i \lambda_i v_i$ for some $\lambda_1, \dots, \lambda_k \in \mathbb{R}$. As $v_i \in V$ and $\lambda_i \in \mathbb{R}$ and V is closed under scalar multiplication we have $\lambda_i v_i \in V$. Now $\lambda_1 v_1$ and $\lambda_2 v_2$ are elements of V, and V is closed under addition, so $\lambda_1 v_1 + \lambda_2 v_2 \in V$. Next, as $\lambda_1 v_1 + \lambda_2 v_2 \in V$ and $\lambda_3 v_3 \in V$ and V is closed under addition we have $\lambda_1 v_1 + \lambda_2 v_2 + \lambda_3 v_3 \in V$. By extending this in the obvious way, we eventually conclude that the vector $w = \lambda_1 v_1 + \dots + \lambda_k v_k$ lies in V as claimed.

□

Dependent lists of length two

Lemma 8.5: Let v and w be vectors in $\mathbb{R}^n$, and suppose that $v \neq 0$ and that the list (v, w) is linearly dependent. Then there is a number α such that $w = \alpha v$.

Proof.
Because the list is dependent, there is a linear relation $\lambda v + \mu w = 0$ where λ and μ are not both zero. There are apparently three possibilities:

(a) $\lambda \neq 0$ and $\mu \neq 0$;
(b) $\lambda = 0$ and $\mu \neq 0$;
(c) $\lambda \neq 0$ and $\mu = 0$.

However, case (c) is not really possible. Indeed, in case (c) the equation $\lambda v + \mu w = 0$ would reduce to $\lambda v = 0$, and we could multiply by λ^{-1} to get $v = 0$; but $v \neq 0$ by assumption. In case (a) or (b) we can take $\alpha = -\lambda/\mu$ and we have $w = \alpha v$. □

Subspaces of $\mathbb{R}^2$

Proposition 19.7: Let V be a subspace of $\mathbb{R}^2$. Then V is either $\{0\}$ or all of $\mathbb{R}^2$ or a straight line through the origin.

Proof.

(a) If $V = \{0\}$ then there is nothing more to say.
(b) Suppose that V contains two vectors v and w such that the list (v, w) is linearly independent. As this is a linearly independent list of two vectors in $\mathbb{R}^2$, it must be a basis. Thus, every vector $x \in \mathbb{R}^2$ is a linear combination of v and w, and therefore lies in V by Proposition 19.6. Thus, we have $V = \mathbb{R}^2$.
(c) Suppose instead that neither (a) nor (b) holds. As (a) does not hold, we can choose a nonzero vector $v \in V$. Let L be the set of all scalar multiples of v, which is a straight line through the origin. As V is a subspace and $v \in V$ we know that every multiple of v lies in V, or in other words that $L \subseteq V$. Now let w be any vector in V. As (b) does not hold, the list (v, w) is linearly dependent, so the Lemma 8.5 tells us that w is a multiple of v and so lies in L. This shows that $V \subseteq L$, so $V = L$.

□

Subspace examples

Consider the set

$$U = \{\begin{bmatrix} w & x & y & z \end{bmatrix}^T \in \mathbb{R}^4 \mid 2w - 4x - 7y + 3z = 1\}.$$

This is not a subspace of $\mathbb{R}^4$. Indeed, as $2 \times 0 - 4 \times 0 - 7 \times 0 + 3 \times 0 \neq 1$ we see that the zero vector is not an element of U. However, a subspace must contain the zero vector, by definition.

Consider the set

$$V = \{\begin{bmatrix} a & b & c \end{bmatrix}^T \in \mathbb{R}^3 \mid a^2 + b^2 = c^2\}.$$

The vectors $u = \begin{bmatrix} 1 & 0 & 1 \end{bmatrix}^T$ and $v = \begin{bmatrix} 0 & 1 & 1 \end{bmatrix}^T$ are elements of V (because $1^2 + 0^2 = 1^2$ and $0^2 + 1^2 = 1^2$) but the vector $u + v = \begin{bmatrix} 1 & 1 & 2 \end{bmatrix}^T$ is not an element of V (because $1^2 + 1^2 \neq 2^2$). This shows that V is not closed under addition, so it is not a subspace of $\mathbb{R}^3$.

Consider the set

$$W = \{\begin{bmatrix} x & y \end{bmatrix}^T \in \mathbb{R}^2 \mid y = \sin(x)\}.$$

This is a subset of $\mathbb{R}^2$ that is not $\{0\}$ or $\mathbb{R}^2$ or a straight line through the origin, so it cannot be a subspace of $\mathbb{R}^2$.

Subspace examples

Consider the set

$$U = \left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} \in \mathbb{R}^3 \mid 10x + 11y + 12z = 0 \right\} = \{u \in \mathbb{R}^3 \mid u.c = 0\} \qquad \left(c = \begin{bmatrix} 10 \\ 11 \\ 12 \end{bmatrix} \right)$$

(a) As $0.c = 0$, we have $0 \in U$.

(b) Suppose that $u, v \in U$, so $u.c = v.c = 0$. Then $(u + v).c = u.c + v.c = 0 + 0 = 0$, so $u + v \in U$. Thus, U is closed under addition.

(c) Suppose that $u \in U$ and $t \in \mathbb{R}$. Then $u.c = 0$, so $(tu).c = t \times 0 = 0$, so $tu \in U$. Thus, U is closed under scalar multiplication.

As U contains zero and is closed under addition and scalar multiplication, it is a subspace of $\mathbb{R}^3$.

Subspace examples

Let V be the set of all vectors in $\mathbb{R}^4$ that have the form

$$v = \begin{bmatrix} 1000s + t & 100s + 10t & 10s + 100t & s + 1000t \end{bmatrix}^T$$

for some $s, t \in \mathbb{R}$.

- Taking $s = t = 0$, we see that the zero vector $v_0 = \begin{bmatrix} 0 & 0 & 0 & 0 \end{bmatrix}^T$ is an element of V.
- Taking $s = 2$ and $t = 4$, we see that the vector $v_1 = \begin{bmatrix} 2004 & 240 & 420 & 4002 \end{bmatrix}^T$ is an element of V.
- Taking $s = 5$ and $t = 1$, we see that the vector $v_2 = \begin{bmatrix} 5001 & 510 & 150 & 1005 \end{bmatrix}^T$ is an element of V.
- Note that $v_1 + v_2 = \begin{bmatrix} 7005 & 750 & 570 & 5007 \end{bmatrix}^T$, which has the required form with $s = 7$ and $t = 5$, so $v_1 + v_2 \in V$. This illustrates (but does not prove) the fact that V is closed under addition.
- Note that $2v_1 = \begin{bmatrix} 4008 & 480 & 840 & 8004 \end{bmatrix}^T$, which has the required form with $s = 4$ and $t = 8$, so $2v_1 \in V$. This illustrates (but does not prove) the fact that V is closed under scalar multiplication.