Lecture 16

Numerical criteria

Corollary: Let V be a d-dimensional subspace of $\mathbb{R}^n$.

(a) Any linearly independent list in V has at most d elements.
(b) Any list that spans V has at least d elements.
(c) Any basis of V has exactly d elements.
(d) Any linearly independent list of length d in V is a basis.
(e) Any list of length d that spans V is a basis.

Proof:

(a) This is just the definition of $\dim(V)$.
(b) Recall: we have inverse functions $\mathbb{R}^d \xrightarrow{\phi} V \xrightarrow{\psi} \mathbb{R}^d$ with $\phi(\lambda) = \sum_i \lambda_i v_i$.
Let $\mathcal{W} = (w_1, \dots, w_r)$ be a list that spans V. We claim that the list $(\psi(w_1), \dots, \psi(w_r))$ spans $\mathbb{R}^d$. Indeed, for any $x \in \mathbb{R}^d$ we have $\phi(x) \in V$, and $\mathcal{W}$ spans V so $\phi(x) = \sum_j \mu_j w_j$ say. We can apply ψ to this to get

$$x = \psi(\phi(x)) = \psi(\sum_j \mu_j w_j) = \sum_j \quad \psi(w_j),$$

which expresses x as a linear combination of the vectors $\psi(w_j)$, as required. We saw earlier that any list that spans $\mathbb{R}^d$ must have length at least d, so $r \geq d$ as claimed.
(c) This holds by combining (a) and (b).
(d) This was proved two slides ago.

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Proof:

(e) Recall: we have inverse functions $\mathbb{R}^d \xrightarrow{\phi} V \xrightarrow{\psi} \mathbb{R}^d$ with $\phi(\lambda) = \sum_i \lambda_i v_i$.
Let $\mathcal{W} = (w_1, \dots, w_d)$ be a list of length d that spans V.
As in (b) we use ϕ and ψ to see that the list $(\psi(w_1), \dots, \psi(w_d))$ spans $\mathbb{R}^d$.
This is a list of length d that spans $\mathbb{R}^d$, so it must be a basis.
In particular, it is linearly independent.
Claim: the original list $\mathcal{W}$ is also linearly independent.
To see this, consider a linear relation $\sum_j \lambda_j w_j = 0$.
By applying ψ to both sides, we get $\sum_i \lambda_j \psi(w_j) = 0$.
As the vectors $\psi(w_j)$ are independent we see that $\qquad$ for all j.
This means that the original relation is the trivial one, as required.
As $\mathcal{W}$ is linearly independent and spans V, it is a basis for V.

Canonical bases

Proposition 20.6: Let V be a subspace of $\mathbb{R}^n$. Then there is a unique RREF matrix B such that the columns of B^T form a basis for V.
(We call this basis the *canonical basis* for V.)

Proof of existence.
Let $\mathcal{U} = (u_1, \dots, u_d)$ be any basis for V, and let A be the matrix with rows $u_1^T, \dots, u_d^T$.

$$A = \left[\begin{array}{c} u_1^T \\ \hline \vdots \\ \hline u_d^T \end{array}\right] \to B = \left[\begin{array}{c} v_1^T \\ \hline \vdots \\ \hline v_d^T \end{array}\right] \qquad B^T = \left[\begin{array}{c|c|c} v_1 & \cdots & v_d \end{array}\right]$$

Let B be the row-reduction of A, let $v_1^T, \dots, v_d^T$ be the rows of B, and put $\mathcal{V} = (v_1, \dots, v_d) =$ the list of columns of B^T. We saw earlier that a row vector can be expressed as a linear combination of the rows of A if and only if it can be expressed as a linear combination of the rows of B. This implies that $\operatorname{span}(\mathcal{V}) = \operatorname{span}(\mathcal{U}) = V$. As $\mathcal{V}$ is a list of length d that spans the d-dimensional space V, we see that $\mathcal{V}$ is actually a basis for V. $\square$

Canonical bases — towards uniqueness

Definition 20.9: Let $x = \begin{bmatrix} x_1 & \cdots & x_n \end{bmatrix}^T$ be a nonzero vector in $\mathbb{R}^n$. We say that x *starts in slot* k if $x_1, \dots, x_{k-1}$ are zero, but x_k is not.

Given a subspace $V \subseteq \mathbb{R}^n$, we say that k is a *jump* for V if there is a nonzero vector $x \in V$ that starts in slot k. We write $J(V)$ for the set of all jumps for V.

Example

- The vector $\begin{bmatrix} 0 & 0 & \mathbf{1} & 11 & 111 \end{bmatrix}^T$ starts in slot 3;
- The vector $\begin{bmatrix} \mathbf{1} & 2 & 3 & 4 & 5 \end{bmatrix}^T$ starts in slot 1;
- The vector $\begin{bmatrix} 0 & 0 & 0 & 0 & \mathbf{0.1234} \end{bmatrix}^T$ starts in .

Examples of jumps

Example: Consider $V = \{\begin{bmatrix} s & -s & t+s & t-s \end{bmatrix}^T \mid s, t \in \mathbb{R}\} \subseteq \mathbb{R}^4$.
If $s \neq 0$ then the vector $x = \begin{bmatrix} s & -s & t+s & t-s \end{bmatrix}^T$ starts in slot 1.
If $s = 0$ but $t \neq 0$ then $x = \begin{bmatrix} 0 & 0 & t & t \end{bmatrix}^T$ and this starts in slot 3.
If $s = t = 0$ then $x = 0$ and x does not start anywhere.
Thus, the possible starting slots for x are 1 and 3, which means that $J(V) = \{1, 3\}$.

Example: Consider the subspace
$W = \{\begin{bmatrix} a & b & c & d & e & f \end{bmatrix}^T \in \mathbb{R}^6 \mid a = b + c = d + e + f = 0\}$.
Any vector $w = \begin{bmatrix} a & b & c & d & e & f \end{bmatrix}^T$ in W can be written as
$w = \begin{bmatrix} 0 & b & -b & d & e & -d-e \end{bmatrix}^T$, where b, d and e are arbitrary.
If $b \neq 0$ then w starts in slot 2.
If $b = 0$ but $d \neq 0$ then $w = \begin{bmatrix} 0 & 0 & 0 & d & e & -d-e \end{bmatrix}^T$ starts in slot 4.
If $b = d = 0$ but $e \neq 0$ then $w = \begin{bmatrix} 0 & 0 & 0 & 0 & e & -e \end{bmatrix}^T$ starts in slot 5.
If $b = d = e = 0$ then $w = 0$ and w does not start anywhere.
Thus, the possible starting slots for w are 2, 4 and 5, so $J(W) = \{2, 4, 5\}$.

Jumps and pivots

Lemma: Let B be an RREF matrix, and suppose that the columns of B^T form a basis for a subspace $V \subseteq \mathbb{R}^n$. Then $J(V) = \{\text{cols of } B \text{ that contain pivots}\}$.

Example proof: Consider $B = \begin{bmatrix} v_1^T \\ \hline v_2^T \\ \hline v_3^T \end{bmatrix} = \begin{bmatrix} 0 & \mathbf{1} & \alpha & 0 & \beta & 0 & \gamma \\ 0 & 0 & 0 & \mathbf{1} & \delta & 0 & \epsilon \\ 0 & 0 & 0 & 0 & 0 & \mathbf{1} & \zeta \end{bmatrix}$.
Put $V = \operatorname{span}(v_1, v_2, v_3) \subseteq \mathbb{R}^7$, so the v_i ($=$ cols of B^T) form a basis for V.
There are pivots in columns 2, 4 and 6, so we must show that $J(V) = \{2, 4, 6\}$. Any $x \in V$ has the form $x = \lambda_1 v_1 + \lambda_2 v_2 + \lambda_3 v_3$

$$= \begin{bmatrix} 0 & \lambda_1 & \lambda_1\alpha_1 & \lambda_2 & \lambda_1\beta + \lambda_2\delta & \lambda_3 & \lambda_1\gamma + \lambda_2\epsilon + \lambda_3\zeta \end{bmatrix}^T.$$

Note that λ_k occurs on its own in the k'th pivot column, and all entries to the left of that involve only $\lambda_1, \dots, \lambda_{k-1}$. Thus, if $\lambda_1, \dots, \lambda_{k-1}$ are all zero but $\lambda_k \neq 0$ then x starts in the k'th pivot column. In more detail:

- If $\lambda_1 \neq 0$ then $x = \begin{bmatrix} 0 & \lambda_1 & * & * & * & * & * \end{bmatrix}^T$ and so starts in slot 2 (the first pivot column).
- If $\lambda_1 = 0 \neq \lambda_2$ then $x = \begin{bmatrix} 0 & 0 & 0 & \lambda_2 & * & * & * \end{bmatrix}^T$ and so x starts in slot 4 (the second pivot column).
- If $\lambda_1 = \lambda_2 = 0 \neq \lambda_3$ then $x = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & \lambda_3 & * \end{bmatrix}^T$ and so x starts in slot 6 (the third pivot column).
- If $\lambda_1 = \lambda_2 = \lambda_3 = 0$ then $x = 0$ and so x does not start anywhere.

Canonical bases — proof of uniqueness

Proposition 20.6: Let V be a subspace of $\mathbb{R}^n$. Then there is a **unique** RREF matrix B such that the columns of B^T form a basis for V.

Sketch proof of uniqueness.
Suppose we have a subspace $V \subseteq \mathbb{R}^n$ and two RREF matrices B and C such that the columns of B^T form a basis for V, and the columns of C^T also form a basis for V. Both B and C must be $d \times n$ matrices, where $d = \dim(V)$. Let $v_1, \dots, v_d$ be the columns of B and let $w_1, \dots, w_d$ be the columns of C. Both B and C have all rows nonzero, and so have d pivots each. The pivot columns are the jumps for V and so are the same for B and C: say columns $p_1, \dots, p_d$.

Now consider one of the vectors v_i. As $v_i \in V$ and $V = \operatorname{span}(w_1, \dots, w_d)$ we can write v_i as a linear combination of the vectors w_j, say $v_i = \lambda_1 w_1 + \cdots + \lambda_d w_d$. By looking in slot p_i we see that $1 = \lambda_i$. By looking in slot p_j (where $j \neq i$) we see that $\lambda_j = 0$. Thus, the sum on the right is just w_i and we get $v_i = w_i$. This holds for all i, so we have $B = C$ as claimed. $\square$

Finding the canonical basis for a span

Method: To find the canonical basis for a subspace $V = \mathrm{span}(v_1, \dots, v_r)$, form the matrix

$$A = \left[\begin{array}{c} v_1^T \\ \hline \vdots \\ \hline v_r^T \end{array}\right]$$

Then row-reduce to get an RREF matrix B, and discard any rows of zeros to get another RREF matrix C. The columns of C^T are the canonical basis for V.

Proof of correctness.

We showed earlier that row operations do not change the span of the rows, and it is clear that discarding rows of zeros does not change the span of the rows either, so the rows of C have the same span as the rows of A. Equivalently, the span of the columns of C^T is the same as the span of the columns of A^T, namely V. Moreover, as each pivot column of C contains a single one, it is easy to see that the rows of C are linearly independent or equivalently the columns of C^T are linearly independent. As they are linearly independent and span V, they form a basis for V. As C is in RREF, this must be the canonical basis. $\square$

Example of finding the canonical basis for a span

Consider again the plane

$$P = \left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} \in \mathbb{R}^3 \mid x + y + z = 0 \right\}.$$

We showed before that $P = \mathrm{span}(u_1, u_2)$, where

$$u_1 = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} \qquad u_2 = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}$$

As the matrix

$$A = \left[\begin{array}{c} u_1^T \\ \hline u_2^T \end{array}\right] = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \end{bmatrix}$$

is already in RREF, we see that the list $\mathcal{U} = (u_1, u_2)$ is the canonical basis for P.

Example of finding the canonical basis for a span

Consider again the subspace

$$V = \{ \begin{bmatrix} w & x & y & z \end{bmatrix}^T \in \mathbb{R}^4 \mid w + 2x + 3y + 4z = 4w + 3x + 2y + z = 0 \}.$$

We showed previously that the vectors

$$c = \begin{bmatrix} -\frac{2}{3} & 1 & 0 & -\frac{1}{3} \end{bmatrix}^T \qquad \text{and} \qquad d = \begin{bmatrix} -\frac{1}{3} & 0 & 1 & -\frac{2}{3} \end{bmatrix}^T.$$

give a (non-canonical) basis for V. To find the canonical basis, we perform the following row-reduction:

$$\left[\begin{array}{c} c^T \\ \hline d^T \end{array}\right] = \begin{bmatrix} -\frac{2}{3} & 1 & 0 & -\frac{1}{3} \\ -\frac{1}{3} & 0 & 1 & -\frac{2}{3} \end{bmatrix} \to \begin{bmatrix} 1 & -\frac{3}{2} & 0 & \frac{1}{2} \\ -\frac{1}{3} & 0 & 1 & -\frac{2}{3} \end{bmatrix} \to$$

$$\begin{bmatrix} 1 & -\frac{3}{2} & 0 & \frac{1}{2} \\ 0 & -\frac{1}{2} & 1 & -\frac{1}{2} \end{bmatrix} \to \begin{bmatrix} 1 & -\frac{3}{2} & 0 & \frac{1}{2} \\ 0 & 1 & -2 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & -3 & 2 \\ 0 & 1 & -2 & 1 \end{bmatrix}$$

We conclude that the vectors $u_1 = \begin{bmatrix} 1 & 0 & -3 & 2 \end{bmatrix}^T$ and $u_2 = \begin{bmatrix} 0 & 1 & -2 & 1 \end{bmatrix}^T$ form the canonical basis for V.