Lecture 17

Finding the canonical basis for an annihilator

Method: Suppose $V = \mathrm{ann}(u_1, \dots, u_r) = \{x \in \mathbb{R}^n \mid x.u_1 = \dots = x.u_r = 0\}$. To find the canonical basis for V:

- Write out the equations $x.u_1 = 0, \dots, x.u_r = 0$, listing the variables in backwards order (x_r down to x_1); then solve by row-reduction.
- Write the general solution as a sum of terms, each of which is an independent variable times a constant vector.
- These constant vectors form the canonical basis for V.

Finding the canonical basis for an annihilator

Example: Put $V = \mathrm{ann}(u_1, u_2, u_3)$, where

$$u_1 = \begin{bmatrix} 9 & 13 & 5 & 3 \end{bmatrix}^T \qquad u_2 = \begin{bmatrix} 1 & 1 & 1 & 1 \end{bmatrix}^T \qquad u_3 = \begin{bmatrix} 7 & 11 & 3 & 1 \end{bmatrix}^T.$$

The equations $x.u_3 = x.u_2 = x.u_1 = 0$ can be written as follows:

$$x_4 + 3x_3 + 11x_2 + 7x_1 = 0 \qquad x_4 + x_3 + x_2 + x_1 = 0 \qquad 3x_4 + 5x_3 + 13x_2 + 9x_1 = 0$$

We can row-reduce the matrix of coefficients as follows:

$$\begin{bmatrix} 1 & 3 & 11 & 7 \\ 1 & 1 & 1 & 1 \\ 3 & 5 & 13 & 9 \end{bmatrix} \to \begin{bmatrix} 0 & 2 & 10 & 6 \\ 1 & 1 & 1 & 1 \\ 0 & 2 & 10 & 6 \end{bmatrix} \to \begin{bmatrix} 0 & 1 & 5 & 3 \\ 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & -4 & -2 \\ 0 & 1 & 5 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

This gives $x_4 - 4x_2 - 2x_1 = x_3 + 5x_2 + 3x_1 = 0$
so $x_4 = 4x_2 + 2x_1$ and $x_3 = -5x_2 - 3x_1$. We thus have

$$x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \\ -5x_2 - 3x_1 \\ 4x_2 + 2x_1 \end{bmatrix} = x_1 \begin{bmatrix} 1 \\ 0 \\ -3 \\ 2 \end{bmatrix} + x_2 \begin{bmatrix} 0 \\ 1 \\ -5 \\ 4 \end{bmatrix}.$$

so $\begin{bmatrix} 1 & 0 & -3 & 2 \end{bmatrix}^T$ and $\begin{bmatrix} 0 & 1 & -5 & 4 \end{bmatrix}^T$ form the canonical basis for V.

Finding the canonical basis for an annihilator

Example: Put $V = \mathrm{ann}(u_1, u_2, u_3)$, where

$$u_1 = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 \end{bmatrix}^T \qquad u_2 = \begin{bmatrix} 1 & 2 & 3 & 3 & 3 \end{bmatrix}^T \qquad u_3 = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \end{bmatrix}^T.$$

To find the canonical basis, write the equations $x.u_3 = x.u_2 = x.u_1 = 0$ as:

$$\begin{aligned} x_5 + x_4 + x_3 + x_2 + x_1 &= 0 \\ 3x_5 + 3x_4 + 3x_3 + 2x_2 + x_1 &= 0 \\ 5x_5 + 4x_4 + 3x_3 + 2x_2 + x_1 &= 0. \end{aligned}$$

We now row-reduce the matrix of coefficients:

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 3 & 3 & 3 & 2 & 1 \\ 5 & 4 & 3 & 2 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & -1 & -2 \\ 0 & -1 & -2 & -3 & -4 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 & 4 \\ 0 & 0 & 0 & 1 & 2 \end{bmatrix}$$

$$\to \begin{bmatrix} 1 & 1 & 1 & 0 & -1 \\ 0 & 1 & 2 & 0 & -2 \\ 0 & 0 & 0 & 1 & 2 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & -1 & 0 & 1 \\ 0 & 1 & 2 & 0 & -2 \\ 0 & 0 & 0 & 1 & 2 \end{bmatrix}$$

This gives $x_5 - x_3 + x_1 = 0$ and $x_4 + 2x_3 - 2x_1 = 0$ and $x_2 + 2x_1 = 0$
so $x_5 = x_3 - x_1$ and $x_4 = -2x_3 + 2x_1$ and $x_2 = -2x_1$, (x_1, x_3 independent)

Finding the canonical basis for an annihilator

$V = \operatorname{ann}(u_1, u_2, u_3)$, where

$$u_1 = \begin{bmatrix}1 & 2 & 3 & 4 & 5\end{bmatrix}^T \qquad u_2 = \begin{bmatrix}1 & 2 & 3 & 3 & 3\end{bmatrix}^T \qquad u_3 = \begin{bmatrix}1 & 1 & 1 & 1 & 1\end{bmatrix}^T.$$

Equations $x.u_3 = x.u_2 = x.u_1 = 0$ give
$x_5 = x_3 - x_1$ and $x_4 = -2x_3 + 2x_1$ and $x_2 = -2x_1$
(with x_1 and x_3 independent).

Thus

$$x = \begin{bmatrix}x_1\\x_2\\x_3\\x_4\\x_5\end{bmatrix} = \begin{bmatrix}x_1\\-2x_1\\x_3\\-2x_3+2x_1\\x_3-x_1\end{bmatrix} = x_1\begin{bmatrix}1\\-2\\0\\2\\-1\end{bmatrix} + x_3\begin{bmatrix}0\\0\\1\\-2\\1\end{bmatrix} = x_1v_1 + x_3v_2 \text{ say.}$$

It follows that the vectors

$$v_1 = \begin{bmatrix}1 & -2 & 0 & 2 & 1\end{bmatrix}^T \qquad \text{and} \qquad v_2 = \begin{bmatrix}0 & 0 & 1 & -2 & 1\end{bmatrix}^T$$

form the canonical basis for V.

Pure matrix method for annihilators

Method: Let A be a $k \times n$ matrix, and let $V \subseteq \mathbb{R}^n$ be the annihilator of the columns of A^T. We can find the canonical basis for V as follows:

(a) Rotate A through 180° to get a matrix A^*.

(b) Row-reduce A^* and discard any rows of zeros to obtain a matrix B^* in RREF. This will have shape $m \times n$ for some m with $m \leq \min(k, n)$.

(c) The matrix B^* will have m pivots (one in each row). Let columns $p_1, \dots, p_m$ be the ones with pivots, and let columns $q_1, \dots, q_{n-m}$ be the ones without pivots.

(d) Delete the pivot columns from B^* to leave an $m \times (n-m)$ matrix, which we call C^*. Let the i'th row of C^* be c_i^T (so $c_i \in \mathbb{R}^{n-m}$ for $1 \leq i \leq m$).

(e) Now construct a new matrix D^* of shape $(n-m) \times n$ as follows: the p_i'th column is $-c_i$, and the q_j'th column is the standard basis vector e_j.

(f) Rotate D^* through 180° to get a matrix D.

(g) The columns of D^T then form the canonical basis for V.

Pure matrix method for annihilators

Example: Again consider $V = \operatorname{ann}(u_1, u_2, u_3)$, where

$$u_1 = \begin{bmatrix}9 & 13 & 5 & 3\end{bmatrix}^T \qquad u_2 = \begin{bmatrix}1 & 1 & 1 & 1\end{bmatrix}^T \qquad u_3 = \begin{bmatrix}7 & 11 & 3 & 1\end{bmatrix}^T.$$

$$A = \begin{bmatrix}u_1^T\\ \hline u_2^T\\ \hline u_3^T\end{bmatrix} = \begin{bmatrix}9 & 13 & 5 & 3\\1 & 1 & 1 & 1\\7 & 11 & 3 & 1\end{bmatrix} \qquad A^* = \begin{bmatrix}1 & 3 & 11 & 7\\1 & 1 & 1 & 1\\3 & 5 & 13 & 9\end{bmatrix}.$$

The matrix A^* is the the matrix of coefficients appearing in our previous approach; as we saw we can row-reduce and delete zeros as follows:

$$A^* = \begin{bmatrix}1 & 3 & 11 & 7\\1 & 1 & 1 & 1\\3 & 5 & 13 & 9\end{bmatrix} \to \begin{bmatrix}1 & 0 & -4 & -2\\0 & 1 & 5 & 3\\0 & 0 & 0 & 0\end{bmatrix} \to \begin{bmatrix}1 & 0 & -4 & -2\\0 & 1 & 5 & 3\end{bmatrix} = B^*.$$

The pivot columns are $p_1 = 1$ and $p_2 = 2$, whereas the non-pivot columns are $q_1 = 3$ and $q_2 = 4$. We now delete the pivot columns to get

$$C^* = \begin{bmatrix}c_1^T\\ \hline c_2^T\end{bmatrix} = \begin{bmatrix}-4 & -2\\5 & 3\end{bmatrix}.$$

$$D^* = \begin{bmatrix}-c_1 & \vert & -c_2 & \vert & e_1 & \vert & e_2\end{bmatrix} = \begin{bmatrix}4 & -5 & 1 & 0\\2 & -3 & 0 & 1\end{bmatrix}; \; D = \begin{bmatrix}1 & 0 & -3 & 2\\0 & 1 & -5 & 4\end{bmatrix}.$$

Canonical basis for V: $\begin{bmatrix}1 & 0 & -3 & 2\end{bmatrix}^T$ and $\begin{bmatrix}0 & 1 & -5 & 4\end{bmatrix}^T$.

Pure matrix method for annihilators

Example: Again consider $V = \operatorname{ann}(u_1, u_2, u_3)$, where

$$u_1 = \begin{bmatrix}1 & 2 & 3 & 4 & 5\end{bmatrix}^T \qquad u_2 = \begin{bmatrix}1 & 2 & 3 & 3 & 3\end{bmatrix}^T \qquad u_3 = \begin{bmatrix}1 & 1 & 1 & 1 & 1\end{bmatrix}^T.$$

$$A = \begin{bmatrix}1 & 2 & 3 & 4 & 5\\1 & 2 & 3 & 3 & 3\\1 & 1 & 1 & 1 & 1\end{bmatrix} \qquad A^* = \begin{bmatrix}1 & 1 & 1 & 1 & 1\\3 & 3 & 3 & 2 & 1\\5 & 4 & 3 & 2 & 1\end{bmatrix}$$

$A^* =$ matrix of coefficients in previous approach. As before:

$$A^* = \begin{bmatrix}1 & 1 & 1 & 1 & 1\\3 & 3 & 3 & 2 & 1\\5 & 4 & 3 & 2 & 1\end{bmatrix} \to \begin{bmatrix}1 & 0 & -1 & 0 & 1\\0 & 1 & 2 & 0 & -2\\0 & 0 & 0 & 1 & 2\end{bmatrix} = B^*.$$

Pivot cols $p_1 = 1$, $p_2 = 2$ and $p_3 = 4$; non-pivot cols $q_1 = 3$ and $q_2 = 5$.

Deleting pivot columns leaves $C^* = \begin{bmatrix}c_1^T\\ \hline c_2^T\\ \hline c_3^T\end{bmatrix} = \begin{bmatrix}-1 & -1\\2 & -2\\0 & 2\end{bmatrix}$

$$D^* = \begin{bmatrix}-c_1 & \vert & -c_2 & \vert & e_1 & \vert & -c_3 & \vert & e_2\end{bmatrix} = \begin{bmatrix}1 & -2 & 1 & 0 & 0\\1 & 2 & 0 & -2 & 1\end{bmatrix}.$$

Rotate: $D = \begin{bmatrix}1 & -2 & 0 & 2 & 1\\0 & 0 & 1 & -2 & 1\end{bmatrix}$. Rows of D give canonical basis for V.

Describing spans as annihilators

We have just discussed a method that finds a basis for an annihilator, and so describes the annihilator as a span.
Opposite problem: describe a span as an annihilator.
In more detail: given $v_1, \dots, v_r$ find $u_1, \dots, u_s$ such that $\text{span}(v_1, \dots, v_r) = \text{ann}(u_1, \dots, u_s)$.

Method:

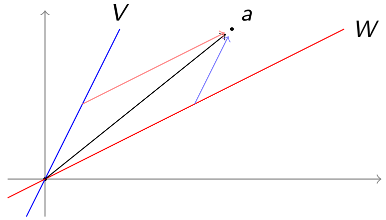
(a) Write out the equations $x.v_r = 0, \dots, x.v_1 = 0$, listing the variables in backwards order (x_r down to x_1).
(b) Solve by row-reduction in the usual way.
(c) Write the general solution as a sum of terms, each of which is an independent variable times a constant vector.
(d) Call these constant vectors $u_1, \dots, u_s$. Then $V = \text{ann}(u_1, \dots, u_s)$.

Sums and intersections of subspaces

Definition: Let V and W be subspaces of $\mathbb{R}^n$. We define

$$V + W = \{x \in \mathbb{R}^n \mid x \text{ can be expressed as } v + w \text{ for some } v \in V \text{ and } w \in W\}$$
$$V \cap W = \{x \in \mathbb{R}^n \mid x \in V \text{ and also } x \in W\}.$$

Example: Put $V = \left\{ \begin{bmatrix} x \\ y \end{bmatrix} \in \mathbb{R}^2 \mid y = 2x \right\}$ $\quad W = \left\{ \begin{bmatrix} x \\ y \end{bmatrix} \in \mathbb{R}^2 \mid 2y = x \right\}$



Then $V \cap W$ is the set of points lying on both lines, but the lines only meet at the origin, so $V \cap W = \{0\}$.
Every point $a \in \mathbb{R}^2$ can be expressed as the sum of a point on V with a point on W, so $V + W = \mathbb{R}^2$.

Sums and intersections of subspaces

Definition: Let V and W be subspaces of $\mathbb{R}^n$. We define

$$V + W = \{x \in \mathbb{R}^n \mid x \text{ can be expressed as } v + w \text{ for some } v \in V \text{ and } w \in W\}$$
$$V \cap W = \{x \in \mathbb{R}^n \mid x \in V \text{ and also } x \in W\}.$$

Example: Put $V = \left\{ \begin{bmatrix} x \\ y \end{bmatrix} \in \mathbb{R}^2 \mid y = 2x \right\}$ $\quad W = \left\{ \begin{bmatrix} x \\ y \end{bmatrix} \in \mathbb{R}^2 \mid 2y = x \right\}$
Algebraically:

- If $\begin{bmatrix} x \\ y \end{bmatrix} \in V \cap W$ then $y = 2x$ and also $x = 2y$, so $x = y = 0$, so $\begin{bmatrix} x \\ y \end{bmatrix} = 0$. Thus $V \cap W = \{0\}$.
- Consider an arbitrary point $a = \begin{bmatrix} x \\ y \end{bmatrix} \in \mathbb{R}^2$. If we put

$$v = \frac{2y - x}{3} \begin{bmatrix} 1 \\ 2 \end{bmatrix} \qquad w = \frac{2x - y}{3} \begin{bmatrix} 2 \\ 1 \end{bmatrix}.$$

we find that $v \in V$ and $w \in W$ and $a = v + w$, which shows that $a \in V + W$. Thus $V + W = \mathbb{R}^2$.

Sum and intersection example

Put

$$V = \{ \begin{bmatrix} w & x & y & z \end{bmatrix}^T \in \mathbb{R}^4 \mid w = y \text{ and } x = z \}$$
$$W = \{ \begin{bmatrix} w & x & y & z \end{bmatrix}^T \in \mathbb{R}^4 \mid w + z = x + y = 0 \}.$$

For a vector $u = \begin{bmatrix} w & x & y & z \end{bmatrix}^T$ to lie in $V \cap W$ we must have $w = y$ and $x = z$ and $w = -z$ and $x = -y$, so $u = \begin{bmatrix} w & -w & w & -w \end{bmatrix}^T$, so $V \cap W$ is just the set of multiples of $\begin{bmatrix} 1 & -1 & 1 & -1 \end{bmatrix}$. Now put

$$U = \{ \begin{bmatrix} w & x & y & z \end{bmatrix}^T \mid w - x - y + z = 0 \} = \text{ann}(\begin{bmatrix} 1 & -1 & -1 & 1 \end{bmatrix}^T).$$

We claim that $V + W = U$. Proof: consider a $u = \begin{bmatrix} w & x & y & z \end{bmatrix}^T$.

- Suppose $u \in V + W$. Then $u = v + w$ for some $v \in V$ and $w \in W$, say $v = \begin{bmatrix} p & q & p & q \end{bmatrix}$ and $w = \begin{bmatrix} -r & -s & s & r \end{bmatrix}^T$. This gives $u = v + w = \begin{bmatrix} p - r & q - s & p + s & q + r \end{bmatrix}$, so

$$w - x - y + z = (p - r) - (q - s) - (p + s) + (q + r) = p - r - q + s - p - s + q + r = 0,$$

proving that $u \in U$ as required.

- Suppose $u \in U$, so $z = x + y - w$. Put $v = \begin{bmatrix} y & x & y & x \end{bmatrix}^T$ and $w = \begin{bmatrix} w - y & 0 & 0 & y - w \end{bmatrix}^T$. We find that $v \in V$ and $w \in W$ and $v + w = u$, which proves that $u \in V + W$ as required.

Sum of spans, intersection of annihilators

Proposition: For lists $v_1, \dots, v_r$ and $w_1, \dots, w_s$ of vectors in $\mathbb{R}^n$, we have

(a) $\text{span}(v_1, \dots, v_r) + \text{span}(w_1, \dots, w_s) = \text{span}(v_1, \dots, v_r, w_1, \dots, w_s)$.

(b) $\text{ann}(v_1, \dots, v_r) \cap \text{ann}(w_1, \dots, w_s) = \text{ann}(v_1, \dots, v_r, w_1, \dots, w_s)$.

Proof.

(a) An arbitrary element $x \in \text{span}(v_1, \dots, v_r) + \text{span}(w_1, \dots, w_s)$ has the form $x = v + w$, where v is an arbitrary element of $\text{span}(v_1, \dots, v_r)$ and w is an arbitrary element of $\text{span}(w_1, \dots, w_s)$. This means that $v = \sum_{i=1}^{r} \lambda_i v_i$ and $w = \sum_{j=1}^{s} \mu_j w_j$ for some coefficients $\lambda_1, \dots, \lambda_r$ and $\mu_1, \dots, \mu_s$, so

$$x = \lambda_1 v_1 + \cdots + \lambda_r v_r + \mu_1 w_1 + \cdots + \mu_s w_s.$$

This is also the general form for an element of $\text{span}(v_1, \dots, v_r, w_1, \dots, w_s)$.

(b) A vector $x \in \mathbb{R}^n$ lies in $\text{ann}(v_1, \dots, v_r)$ if and only if $x.v_1 = \cdots = x.v_r = 0$. Similarly, x lies in $\text{ann}(w_1, \dots, w_s)$ iff $x.w_1 = \cdots = x.w_s$. Thus, x lies in $\text{ann}(v_1, \dots, v_r) \cap \text{ann}(w_1, \cdots, w_s)$ iff both sets of equations are satisfied, or in other words $x.v_1 = \cdots = x.v_r = x.w_1 = \cdots = x.w_s = 0$.
This is precisely the condition for x to lie in $\text{ann}(v_1, \dots, v_r, w_1, \dots, w_s)$.

□

The dimension formula

Dimensions of V, W, $V \cap W$ and $V + W$ are linked by the following formula:

$$\dim(V \cap W) + \dim(V + W) = \dim(V) + \dim(W).$$

Example:

$$\begin{aligned}
V &= \text{span}(e_1, \dots, e_p, e_{p+1}, \dots, e_{p+q}) \\
W &= \text{span}(e_1, \dots, e_p, e_{p+q+1}, \dots, e_{p+q+r}) \\
V \cap W &= \text{span}(e_1, \dots, e_p) \\
V + W &= \text{span}(e_1, \dots, e_{p+q+r}) \\
\dim(V \cap W) + \dim(V + W) &= p + (p + q + r) = 2p + q + r \\
\dim(V) + \dim(W) &= (p + q) + (p + r) = 2p + q + r.
\end{aligned}$$

- If we know three of the numbers $\dim(V \cap W)$, $\dim(V + W)$, $\dim(V)$ and $\dim(W)$, we can rearrange the formula to find the fourth.
- If you believe that you have found bases for V, W, $V \cap W$ and $V + W$, you can use the formula as a check that your bases are correct.

We will not prove the formula, except to say that one can choose bases to make the proof like the above example. Details would be a digression.