

Method 21.5: To find the sum of two subspaces $V, W \subseteq \mathbb{R}^n$:

- (a) Find a list \mathcal{V} such that $V = \text{span}(\mathcal{V})$. (If V is given as an annihilator, use earlier method to find the canonical basis \mathcal{V} for V ; then $V = \text{span}(\mathcal{V})$.)
- (b) Find a list \mathcal{W} such that $W = \text{span}(\mathcal{W})$ (in the same way).
- (c) Now $V + W$ is the span of the combined list \mathcal{V}, \mathcal{W} . If desired, we can make this list the rows of a matrix then row-reduce and discard zeros to get the canonical basis for $V + W$.

Method 21.6: To find the intersection of two subspaces $V, W \subseteq \mathbb{R}^n$:

- (a) Find a list \mathcal{V}' such that $V = \text{ann}(\mathcal{V}')$. It may be that V is given to us as the annihilator of some list, in which case there is nothing to do. Alternatively, if V is given to us as the span of some list, then gave a method earlier to find a list \mathcal{V}' such that $\text{ann}(\mathcal{V}') = V$.
- (b) Find a list \mathcal{W}' such that $W = \text{ann}(\mathcal{W}')$ (in the same way).
- (c) Now $V \cap W$ is the annihilator of the combined list $\mathcal{V}', \mathcal{W}'$. Earlier we described how to find the canonical basis for an annihilator, so we can use that to get the canonical basis for $V \cap W$.

Sum and intersection example

Put $V = \text{span}(v_1, v_2, v_3)$ and $W = \text{span}(w_1, w_2, w_3)$ where

$$v_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \quad v_2 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \end{bmatrix} \quad v_3 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \quad w_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} \quad w_2 = \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \end{bmatrix} \quad w_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix}$$

Claim: $V + W = \mathbb{R}^4$. Systematic proof: recall

$V + W = \text{span}(v_1, v_2, v_3, w_1, w_2, w_3)$ and row-reduce:

$$\begin{bmatrix} v_1^T \\ v_2^T \\ v_3^T \\ w_1^T \\ w_2^T \\ w_3^T \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} e_1^T \\ e_2^T \\ e_3^T \\ e_4^T \\ 0 \\ 0 \end{bmatrix}$$

Conclusion: (e_1, e_2, e_3, e_4) is the canonical basis for $V + W$, so $V + W = \mathbb{R}^4$.

More efficiently:

$$e_1 = v_1 \quad e_2 = w_1 - v_1 \quad e_3 = v_2 - w_1 \quad e_4 = v_3 - v_2.$$

It follows that e_1, e_2, e_3 and e_4 are all in $V + W$, so $V + W = \mathbb{R}^4$.

Sum and intersection example

$$v_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \quad v_2 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \end{bmatrix} \quad v_3 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \quad w_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} \quad w_2 = \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \end{bmatrix} \quad w_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix}$$

Now find $V \cap W$. First step: describe V as an annihilator. Write equations $x \cdot v_3 = 0, x \cdot v_2 = 0$ and $x \cdot v_1 = 0$, with the variables x_i in descending order:

$$\begin{aligned} x_4 + x_3 + x_2 + x_1 &= 0 \\ x_3 + x_2 + x_1 &= 0 \\ x_1 &= 0. \end{aligned}$$

Clearly $x_1 = x_4 = 0$ and $x_3 = -x_2$, with x_2 arbitrary. Thus:

$$x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ x_2 \\ -x_2 \\ 0 \end{bmatrix} = x_2 \begin{bmatrix} 0 \\ 1 \\ -1 \\ 0 \end{bmatrix}$$

We conclude that $V = \text{ann}(a)$, where $a = [0 \ 1 \ -1 \ 0]^T$.

Sum and intersection example

$$v_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \quad v_2 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \end{bmatrix} \quad v_3 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \quad w_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} \quad w_2 = \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \end{bmatrix} \quad w_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix}$$

Next step: describe W as an annihilator. Write down the equations $x \cdot w_3 = 0$, $x \cdot w_2 = 0$ and $x \cdot w_1 = 0$, with the variables x_i in descending order:

$$\begin{aligned} x_4 + x_3 &= 0 \\ x_3 + x_2 &= 0 \\ x_2 + x_1 &= 0. \end{aligned}$$

This easily gives $x_4 = -x_3 = x_2 = -x_1$, so

$$x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} x_1 \\ -x_1 \\ x_1 \\ -x_1 \end{bmatrix} = x_1 \begin{bmatrix} 1 \\ -1 \\ 1 \\ -1 \end{bmatrix}$$

We conclude that $W = \text{ann}(b)$, where $b = [1 \ -1 \ 1 \ -1]^T$.

Sum and intersection example

$$a = [0 \ 1 \ -1 \ 0]^T \quad b = [1 \ -1 \ 1 \ -1]^T$$

We now have $V = \text{ann}(a)$ and $W = \text{ann}(b)$ so $V \cap W = \text{ann}(a, b)$. To find the canonical basis for this, we write the equations $x \cdot b = 0$ and $x \cdot a = 0$, again with the variables in decreasing order:

$$\begin{aligned} -x_4 + x_3 - x_2 + x_1 &= 0 \\ -x_3 + x_2 &= 0 \end{aligned}$$

After row-reduction we get $x_4 = x_1$ and $x_3 = x_2$ with x_1 and x_2 arbitrary. This gives

$$x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \\ x_2 \\ x_1 \end{bmatrix} = x_1 \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix} + x_2 \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \end{bmatrix}.$$

We conclude that the vectors $u_1 = [1 \ 0 \ 0 \ 1]^T$ and $u_2 = [0 \ 1 \ 1 \ 0]^T$ form the canonical basis for $V \cap W$. As a sanity check we can note that

$$\begin{aligned} u_1 &= v_1 - v_2 + v_3 \in V & u_2 &= v_2 - v_1 \in V \\ u_1 &= w_1 - w_2 + w_3 \in W & u_2 &= w_2 \in W. \end{aligned}$$

These equations show directly that u_1 and u_2 lie in $V \cap W$.

Dimension check

We will use the dimension formula to check our calculation.

- ▶ $V = \text{span}(v_1, v_2, v_3)$ and one can check that this list is independent so $\dim(V) = 3$.
- ▶ $W = \text{span}(w_1, w_2, w_3)$ and one can check that this list is independent so $\dim(W) = 3$.
- ▶ We showed that $V + W = \mathbb{R}^4$ so $\dim(V + W) = 4$.
- ▶ We showed that u_1, u_2 is a basis for $V \cap W$ so $\dim(V \cap W) = 2$.
- ▶ Now $\dim(V + W) + \dim(V \cap W) = 4 + 2 = 6$ and $\dim(V) + \dim(W) = 3 + 3 = 6$. As expected, these are the same.

Sum and intersection example

Put $V = \text{span}(v_1, v_2)$ and $W = \text{span}(w_1, w_2)$ where

$$v_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \quad v_2 = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix} \quad w_1 = \begin{bmatrix} -3 \\ -1 \\ 1 \\ 3 \end{bmatrix} \quad w_2 = \begin{bmatrix} 0 \\ 1 \\ -1 \\ 0 \end{bmatrix}$$

We will find the canonical bases for V , W , $V + W$ and $V \cap W$. For V :

$$\left[\begin{array}{c} v_1^T \\ v_2^T \end{array} \right] = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & -1 & -2 \\ 0 & 1 & 2 & 3 \end{bmatrix}$$

Thus: the vectors $v'_1 = [1 \ 0 \ -1 \ -2]^T$ and $v'_2 = [0 \ 1 \ 2 \ 3]^T$ form the canonical basis for V .

Similarly, the row-reduction

$$\left[\begin{array}{c} w_1^T \\ w_2^T \end{array} \right] = \begin{bmatrix} -3 & -1 & 1 & 3 \\ 0 & 1 & -1 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} -3 & 0 & 0 & 3 \\ 0 & 1 & -1 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & -1 & 0 \end{bmatrix}$$

shows that the vectors $w'_1 = [1 \ 0 \ 0 \ -1]^T$ and $w'_2 = [0 \ 1 \ -1 \ 0]^T$ form the canonical basis for W .

Sum and intersection example

$V = \text{span}(v'_1, v'_2)$ and $W = \text{span}(w'_1, w'_2)$ where

$$v'_1 = \begin{bmatrix} 1 \\ 0 \\ -1 \\ -2 \end{bmatrix} \quad v'_2 = \begin{bmatrix} 0 \\ 1 \\ 2 \\ 3 \end{bmatrix} \quad w'_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ -1 \end{bmatrix} \quad w'_2 = \begin{bmatrix} 0 \\ 1 \\ -1 \\ 0 \end{bmatrix}$$

Next find the canonical basis for $V + W = \text{span}(v'_1, v'_2, w'_1, w'_2)$, by row-reducing either the matrix $[v'_1 | v'_2 | w'_1 | w'_2]^T$:

$$\begin{bmatrix} 1 & 0 & -1 & -2 \\ 0 & 1 & 2 & 3 \\ 1 & 0 & 0 & -1 \\ 0 & 1 & -1 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & -1 & -2 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & -3 & -3 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

We conclude that the following vectors form the canonical basis for $V + W$:

$$u_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ -1 \end{bmatrix} \quad u_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix} \quad u_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix}.$$

In particular, we have $\dim(V + W) = 3$.

Sum and intersection example

$$V = \text{span}(v'_1, v'_2) \quad v'_1 = [1 \ 0 \ -1 \ -2]^T \quad v'_2 = [0 \ 1 \ 2 \ 3]^T$$

$$W = \text{span}(w'_1, w'_2) \quad w'_1 = [1 \ 0 \ 0 \ -1]^T \quad w'_2 = [0 \ 1 \ -1 \ 0]^T$$

Next, to understand $V \cap W$, we need to write V and W as annihilators.

For W : put $b_1 = [1 \ 0 \ 0 \ 1]^T$ and $b_2 = [0 \ 1 \ 1 \ 0]^T$.

After considering the form of the vectors w'_1 and w'_2 we see that

$$W = \left\{ \begin{bmatrix} x_1 \\ x_2 \\ -x_2 \\ -x_1 \end{bmatrix} \mid x_1, x_2 \in \mathbb{R} \right\} = \left\{ \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} \mid x_1 + x_4 = x_2 + x_3 = 0 \right\} = \text{ann}(b_1, b_2).$$

For V : the equations $x \cdot v'_1 = 0$ and $x \cdot v'_2 = 0$ are $-2x_4 - x_3 + x_1 = 0$ and $3x_4 + 2x_3 + x_2 = 0$. Solution:

$$x_3 = -2x_2 - 3x_1 \quad x_4 = x_2 + 2x_1 \quad (x_2 \text{ and } x_1 \text{ arbitrary})$$

$$x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \\ -2x_2 - 3x_1 \\ x_2 + 2x_1 \end{bmatrix} = x_1 \begin{bmatrix} 1 \\ 0 \\ -3 \\ 2 \end{bmatrix} + x_2 \begin{bmatrix} 0 \\ 1 \\ -2 \\ 1 \end{bmatrix} = x_1 a_1 + x_2 a_2 \text{ say.}$$

Thus $V = \text{ann}(a_1, a_2)$, where $a_1 = [1 \ 0 \ -3 \ 2]^T$, $a_2 = [0 \ 1 \ -2 \ 1]^T$.

Sum and intersection example

$$a_1 = [1 \ 0 \ -3 \ 2]^T \quad a_2 = [0 \ 1 \ -2 \ 1]^T$$

$$b_1 = [1 \ 0 \ 0 \ 1]^T \quad b_2 = [0 \ 1 \ 1 \ 0]^T$$

We now have

$$V \cap W = \text{ann}(a_1, a_2) \cap \text{ann}(b_1, b_2) = \text{ann}(a_1, a_2, b_1, b_2).$$

To find the canonical basis, solve $x \cdot b_2 = x \cdot b_1 = x \cdot a_2 = x \cdot a_1 = 0$:

$$\begin{aligned} x_3 + x_2 &= 0 & x_4 + x_1 &= 0 \\ x_4 - 2x_3 + x_2 &= 0 & 2x_4 - 3x_3 + x_1 &= 0. \end{aligned}$$

The first two equations give $x_3 = -x_2$ and $x_4 = -x_1$, which we can substitute into the remaining equations to get $x_2 = x_1/3$. This leads to

$x = x_1 [1 \ 1/3 \ -1/3 \ -1]^T$, so the vector $c = [1 \ 1/3 \ -1/3 \ -1]^T$ is

(by itself) the canonical basis for $V \cap W$. In particular, we have

$\dim(V \cap W) = 1$.

As a check, we note that

$$\dim(V + W) + \dim(V \cap W) = 3 + 1 = 2 + 2 = \dim(V) + \dim(W),$$

as expected.