Lecture 18

Finding sums and intersections

Method 21.5: To find the sum of two subspaces $V, W \subseteq \mathbb{R}^n$:

(a) Find a list $\mathcal{V}$ such that $V = \operatorname{span}(\mathcal{V})$. (If V is given as an annihilator, use earlier method to find the canonical basis $\mathcal{V}$ for V; then $V = \operatorname{span}(\mathcal{V})$.)

(b) Find a list $\mathcal{W}$ such that $W = \operatorname{span}(\mathcal{W})$ (in the same way).

(c) Now $V + W$ is the span of the combined list $\mathcal{V}, \mathcal{W}$. If desired, we can make this list the rows of a matrix then row-reduce and discard zeros to get the canonical basis for $V + W$.

Method 21.6: To find the intersection of two subspaces $V, W \subseteq \mathbb{R}^n$:

(a) Find a list $\mathcal{V}'$ such that $V = \operatorname{ann}(\mathcal{V}')$. It may be that V is given to us as the annihilator of some list, in which case there is nothing to do. Alternatively, if V is given to as as the span of some list, then gave a method earlier to find a list $\mathcal{V}'$ such that $\operatorname{ann}(\mathcal{V}') = V$.

(b) Find a list $\mathcal{W}'$ such that $W = \operatorname{ann}(\mathcal{W}')$ (in the same way).

(c) Now $V \cap W$ is the annihilator of the combined list $\mathcal{V}', \mathcal{W}'$. Earlier we described how to find the canonical basis for an annihilator, so we can use that to get the canonical basis for $V \cap W$.

Sum and intersection example

Put $V = \operatorname{span}(v_1, v_2, v_3)$ and $W = \operatorname{span}(w_1, w_2, w_3)$ where

$$v_1 = \begin{bmatrix}1\\0\\0\\0\end{bmatrix} \quad v_2 = \begin{bmatrix}1\\1\\1\\0\end{bmatrix} \quad v_3 = \begin{bmatrix}1\\1\\1\\1\end{bmatrix} \quad w_1 = \begin{bmatrix}1\\1\\0\\0\end{bmatrix} \quad w_2 = \begin{bmatrix}0\\1\\1\\0\end{bmatrix} \quad w_3 = \begin{bmatrix}0\\0\\1\\1\end{bmatrix}$$

Claim: $V + W = \mathbb{R}^4$. Systematic proof: recall $V + W = \operatorname{span}(v_1, v_2, v_3, w_1, w_2, w_3))$ and row-reduce:

$$\begin{bmatrix} v_1^T \\ \hline v_2^T \\ \hline v_3^T \\ \hline w_1^T \\ \hline w_2^T \\ \hline w_3^T \end{bmatrix} = \begin{bmatrix} 1&0&0&0\\ 1&1&1&0\\ 1&1&1&1\\ 1&1&0&0\\ 0&1&1&0\\ 0&0&1&1 \end{bmatrix} \to \begin{bmatrix} 1&0&0&0\\ 0&1&0&0\\ 0&0&1&0\\ 0&0&0&1\\ \hline 0&0&0&0\\ 0&0&0&0 \end{bmatrix} = \begin{bmatrix} e_1^T \\ \hline e_2^T \\ \hline e_3^T \\ \hline e_4^T \\ \hline 0 \\ \hline 0 \end{bmatrix}$$

Conclusion: (e_1, e_2, e_3, e_4) is the canonical basis for $V + W$, so $V + W = \mathbb{R}^4$.
More efficiently:

$$e_1 = v_1 \qquad e_2 = w_1 - v_1 \qquad e_3 = v_2 - w_1 \qquad e_4 = v_3 - v_2.$$

It follows that e_1, e_2, e_3 and e_4 are all in $V + W$, so $V + W = \mathbb{R}^4$.

Sum and intersection example

$$v_1 = \begin{bmatrix}1\\0\\0\\0\end{bmatrix} \quad v_2 = \begin{bmatrix}1\\1\\1\\0\end{bmatrix} \quad v_3 = \begin{bmatrix}1\\1\\1\\1\end{bmatrix} \quad w_1 = \begin{bmatrix}1\\1\\0\\0\end{bmatrix} \quad w_2 = \begin{bmatrix}0\\1\\1\\0\end{bmatrix} \quad w_3 = \begin{bmatrix}0\\0\\1\\1\end{bmatrix}$$

Now find $V \cap W$. First step: describe V as an annihilator. Write equations $x.v_3 = 0$, $x.v_2 = 0$ and $x.v_1 = 0$, with the variables x_i in descending order:

$$\begin{aligned} x_4 + x_3 + x_2 + x_1 &= 0 \\ x_3 + x_2 + x_1 &= 0 \\ x_1 &= 0. \end{aligned}$$

Clearly $x_1 = x_4 = 0$ and $x_3 = -x_2$, with x_2 arbitrary. Thus:

$$x = \begin{bmatrix}x_1\\x_2\\x_3\\x_4\end{bmatrix} = \begin{bmatrix}0\\x_2\\-x_2\\0\end{bmatrix} = x_2 \begin{bmatrix}0\\1\\-1\\0\end{bmatrix}$$

We conclude that $V = \operatorname{ann}(a)$, where $a = \begin{bmatrix}0 & 1 & -1 & 0\end{bmatrix}^T$.

Sum and intersection example

$$v_1 = \begin{bmatrix}1\\0\\0\\0\end{bmatrix} \quad v_2 = \begin{bmatrix}1\\1\\1\\0\end{bmatrix} \quad v_3 = \begin{bmatrix}1\\1\\1\\1\end{bmatrix} \quad w_1 = \begin{bmatrix}1\\1\\0\\0\end{bmatrix} \quad w_2 = \begin{bmatrix}0\\1\\1\\0\end{bmatrix} \quad w_3 = \begin{bmatrix}0\\0\\1\\1\end{bmatrix}$$

Next step: describe W as an annihilator. Write down the equations $x.w_3 = 0$, $x.w_2 = 0$ and $x.w_1 = 0$, with the variables x_i in descending order:

$$\begin{aligned} x_4 + x_3 &= 0 \\ x_3 + x_2 &= 0 \\ x_2 + x_1 &= 0. \end{aligned}$$

This easily gives $x_4 = -x_3 = x_2 = -x_1$, so

$$x = \begin{bmatrix}x_1\\x_2\\x_3\\x_4\end{bmatrix} = \begin{bmatrix}x_1\\-x_1\\x_1\\-x_1\end{bmatrix} = x_1 \begin{bmatrix}1\\-1\\1\\-1\end{bmatrix}$$

We conclude that $W = \text{ann}(b)$, where $b = \begin{bmatrix}1 & -1 & 1 & -1\end{bmatrix}^T$.

Sum and intersection example

$$a = \begin{bmatrix}0 & 1 & -1 & 0\end{bmatrix}^T \qquad b = \begin{bmatrix}1 & -1 & 1 & -1\end{bmatrix}^T$$

We now have $V = \text{ann}(a)$ and $W = \text{ann}(b)$ so $V \cap W = \text{ann}(a, b)$. To find the canonical basis for this, we write the equations $x.b = 0$ and $x.a = 0$, again with the variables in decreasing order:

$$\begin{aligned} -x_4 + x_3 - x_2 + x_1 &= 0 \\ -x_3 + x_2 &= 0 \end{aligned}$$

After row-reduction we get $x_4 = x_1$ and $x_3 = x_2$ with x_1 and x_2 arbitrary. This gives

$$x = \begin{bmatrix}x_1\\x_2\\x_3\\x_4\end{bmatrix} = \begin{bmatrix}x_1\\x_2\\x_2\\x_1\end{bmatrix} = x_1 \begin{bmatrix}1\\0\\0\\1\end{bmatrix} + x_2 \begin{bmatrix}0\\1\\1\\0\end{bmatrix}.$$

We conclude that the vectors $u_1 = \begin{bmatrix}1 & 0 & 0 & 1\end{bmatrix}^T$ and $u_2 = \begin{bmatrix}0 & 1 & 1 & 0\end{bmatrix}^T$ form the canonical basis for $V \cap W$. As a sanity check we can note that

$$\begin{aligned} u_1 &= v_1 - v_2 + v_3 \in V & u_2 &= v_2 - v_1 \in V \\ u_1 &= w_1 - w_2 + w_3 \in W & u_2 &= w_2 \in W. \end{aligned}$$

These equations show directly that u_1 and u_2 lie in $V \cap W$.

Dimension check

We will use the dimension formula to check our calculation.

- $V = \text{span}(v_1, v_2, v_3)$ and one can check that this list is independent so $\dim(V) = 3$.
- $W = \text{span}(w_1, w_2, w_3)$ and one can check that this list is independent so $\dim(W) = 3$.
- We showed that $V + W = \mathbb{R}^4$ so $\dim(V + W) = 4$.
- We showed that u_1, u_2 is a basis for $V \cap W$ so $\dim(V \cap W) = 2$.
- Now $\dim(V + W) + \dim(V \cap W) = 4 + 2 = 6$ and $\dim(V) + \dim(W) = 3 + 3 = 6$. As expected, these are the same.

Sum and intersection example

Put $V = \text{span}(v_1, v_2)$ and $W = \text{span}(w_1, w_2)$ where

$$v_1 = \begin{bmatrix}1\\1\\1\\1\end{bmatrix} \quad v_2 = \begin{bmatrix}1\\2\\3\\4\end{bmatrix} \quad w_1 = \begin{bmatrix}-3\\-1\\1\\3\end{bmatrix} \quad w_2 = \begin{bmatrix}0\\1\\-1\\0\end{bmatrix}$$

We will find the canonical bases for V, W, $V + W$ and $V \cap W$. For V:

$$\left[\begin{array}{c} v_1^T \\ \hline v_2^T \end{array}\right] = \begin{bmatrix}1 & 1 & 1 & 1\\1 & 2 & 3 & 4\end{bmatrix} \to \begin{bmatrix}1 & 1 & 1 & 1\\0 & 1 & 2 & 3\end{bmatrix} \to \begin{bmatrix}1 & 0 & -1 & -2\\0 & 1 & 2 & 3\end{bmatrix}$$

Thus: the vectors $v_1' = \begin{bmatrix}1 & 0 & -1 & -2\end{bmatrix}^T$ and $v_2' = \begin{bmatrix}0 & 1 & 2 & 3\end{bmatrix}^T$ form the canonical basis for V.
Similarly, the row-reduction

$$\left[\begin{array}{c} w_1^T \\ \hline w_2^T \end{array}\right] = \begin{bmatrix}-3 & -1 & 1 & 3\\0 & 1 & -1 & 0\end{bmatrix} \to \begin{bmatrix}-3 & 0 & 0 & 3\\0 & 1 & -1 & 0\end{bmatrix} \to \begin{bmatrix}1 & 0 & 0 & -1\\0 & 1 & -1 & 0\end{bmatrix}$$

shows that the vectors $w_1' = \begin{bmatrix}1 & 0 & 0 & -1\end{bmatrix}^T$ and $w_2' = \begin{bmatrix}0 & 1 & -1 & 0\end{bmatrix}^T$ form the canonical basis for W.

Sum and intersection example

$V = \mathrm{span}(v_1', v_2')$ and $W = \mathrm{span}(w_1', w_2')$ where

$$v_1' = \begin{bmatrix} 1 \\ 0 \\ -1 \\ -2 \end{bmatrix} \qquad v_2' = \begin{bmatrix} 0 \\ 1 \\ 2 \\ 3 \end{bmatrix} \qquad w_1' = \begin{bmatrix} 1 \\ 0 \\ 0 \\ -1 \end{bmatrix} \qquad w_2' = \begin{bmatrix} 0 \\ 1 \\ -1 \\ 0 \end{bmatrix}$$

Next find the canonical basis for $V + W = \mathrm{span}(v_1', v_2', w_1', w_2')$, by row-reducing either the matrix $[v_1'|v_2'|w_1'|w_2']^T$:

$$\begin{bmatrix} 1 & 0 & -1 & -2 \\ 0 & 1 & 2 & 3 \\ 1 & 0 & 0 & -1 \\ 0 & 1 & -1 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & -1 & -2 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & -3 & -3 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

We conclude that the following vectors form the canonical basis for $V + W$:

$$u_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ -1 \end{bmatrix} \qquad u_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix} \qquad u_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix}.$$

In particular, we have $\dim(V + W) = 3$.

Sum and intersection example

$$V = \mathrm{span}(v_1', v_2') \qquad v_1' = \begin{bmatrix} 1 & 0 & -1 & -2 \end{bmatrix}^T \qquad v_2' = \begin{bmatrix} 0 & 1 & 2 & 3 \end{bmatrix}^T$$

$$W = \mathrm{span}(w_1', w_2') \qquad w_1' = \begin{bmatrix} 1 & 0 & 0 & -1 \end{bmatrix}^T \qquad w_2' = \begin{bmatrix} 0 & 1 & -1 & 0 \end{bmatrix}^T$$

Next, to understand $V \cap W$, we need to write V and W as annihilators.
For W: put $b_1 = \begin{bmatrix} 1 & 0 & 0 & 1 \end{bmatrix}^T$ and $b_2 = \begin{bmatrix} 0 & 1 & 1 & 0 \end{bmatrix}^T$.
After considering the form of the vectors w_1' and w_2' we see that

$$W = \left\{ \begin{bmatrix} x_1 \\ x_2 \\ -x_2 \\ -x_1 \end{bmatrix} \mid x_1, x_2 \in \mathbb{R} \right\} = \left\{ \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} \mid x_1 + x_4 = x_2 + x_3 = 0 \right\} = \mathrm{ann}(b_1, b_2).$$

For V: the equations $x.v_1' = 0$ and $x.v_2' = 0$ are $-2x_4 - x_3 + x_1 = 0$ and $3x_4 + 2x_3 + x_2 = 0$. Solution:

$$x_3 = -2x_2 - 3x_1 \qquad x_4 = x_2 + 2x_1 \qquad (\ x_2 \text{ and } x_1 \text{ arbitrary})$$

$$x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \\ -2x_2 - 3x_1 \\ x_2 + 2x_1 \end{bmatrix} = x_1 \begin{bmatrix} 1 \\ 0 \\ -3 \\ 2 \end{bmatrix} + x_2 \begin{bmatrix} 0 \\ 1 \\ -2 \\ 1 \end{bmatrix} = x_1 a_1 + x_2 a_2 \text{ say.}$$

Thus $V = \mathrm{ann}(a_1, a_2)$, where $a_1 = \begin{bmatrix} 1 & 0 & -3 & 2 \end{bmatrix}^T$, $a_2 = \begin{bmatrix} 0 & 1 & -2 & 1 \end{bmatrix}^T$.

Sum and intersection example

$$a_1 = \begin{bmatrix} 1 & 0 & -3 & 2 \end{bmatrix}^T \qquad a_2 = \begin{bmatrix} 0 & 1 & -2 & 1 \end{bmatrix}^T$$
$$b_1 = \begin{bmatrix} 1 & 0 & 0 & 1 \end{bmatrix}^T \qquad b_2 = \begin{bmatrix} 0 & 1 & 1 & 0 \end{bmatrix}^T$$

We now have

$$V \cap W = \mathrm{ann}(a_1, a_2) \cap \mathrm{ann}(b_1, b_2) = \mathrm{ann}(a_1, a_2, b_1, b_2).$$

To find the canonical basis, solve $x.b_2 = x.b_1 = x.a_2 = x.a_1 = 0$:

$$x_3 + x_2 = 0 \qquad x_4 + x_1 = 0$$
$$x_4 - 2x_3 + x_2 = 0 \qquad 2x_4 - 3x_3 + x_1 = 0.$$

The first two equations give $x_3 = -x_2$ and $x_4 = -x_1$, which we can substitute into the remaining equations to get $x_2 = x_1/3$. This leads to $x = x_1 \begin{bmatrix} 1 & 1/3 & -1/3 & -1 \end{bmatrix}^T$, so the vector $c = \begin{bmatrix} 1 & 1/3 & -1/3 & -1 \end{bmatrix}^T$ is (by itself) the canonical basis for $V \cap W$. In particular, we have $\dim(V \cap W) = 1$.
As a check, we note that

$$\dim(V + W) + \dim(V \cap W) = 3 + 1 = 2 + 2 = \dim(V) + \dim(W),$$

as expected.