Lecture 21

Linear and quadratic forms

Definition 23.19:

(a) A *linear form* on $\mathbb{R}^n$ is a function of the form $L(x) = \sum_{i=1}^n a_i x_i$ (for some constants $a_1, \ldots, a_n$).

(b) A *quadratic form* on $\mathbb{R}^n$ is a function of the form $Q(x) = \sum_{i=1}^n \sum_{j=1}^n b_{ij} x_i x_j$ (for some constants b_{ij}).

Example 23.20:

(a) We can define a linear form on $\mathbb{R}^3$ by $L(x) = 7x_1 - 8x_2 + 9x_3$.

(b) We can define a quadratic form on $\mathbb{R}^4$ by $Q(x) = 10x_1x_2 + 12x_3x_4 - 14x_1x_4 - 16x_2x_3$.

Given a linear form $L(x) = \sum_i a_i x_i$, we can form the vector $a = \begin{bmatrix} a_1 & \cdots & a_n \end{bmatrix}^T$, and clearly $L(x) = a.x = a^T x$.

Symmetric expressions for quadratic forms

Consider a quadratic form $Q(x) = \sum_{i,j} b_{ij} x_i x_j$.
Form the matrix B with entries b_{ij}: we find that $Q(x) = x^T B x$.
For example, if $n = 2$ and $Q(x) = 2x_1^2 + 4x_1x_2 + 7x_2^2$ then $B = \begin{bmatrix} 2 & 4 \\ 0 & 7 \end{bmatrix}$ and

$$x^T B x = \begin{bmatrix} x_1 & x_2 \end{bmatrix} \begin{bmatrix} 2 & 4 \\ 0 & 7 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_1 & x_2 \end{bmatrix} \begin{bmatrix} 2x_1 + 4x_2 \\ 7x_2 \end{bmatrix} = 2x_1^2 + 4x_1x_2 + 7x_2^2 = Q(x).$$

Alternatively, note that $x_1x_2 = x_2x_1$:

(a) Rewriting the same $Q(x)$ as $2x_1^2 + 3x_1x_2 + 1x_2x_1 + 7x_2^2$ gives $B = \begin{bmatrix} 2 & 3 \\ 1 & 7 \end{bmatrix}$.

(b) Rewriting the same $Q(x)$ as $2x_1^2 + 2x_1x_2 + 2x_2x_1 + 7x_2^2$ gives $B = \begin{bmatrix} 2 & 2 \\ 2 & 7 \end{bmatrix}$.

(c) Rewriting the same $Q(x)$ as $2x_1^2 + 1x_1x_2 + 3x_2x_1 + 7x_2^2$ gives $B = \begin{bmatrix} 2 & 1 \\ 3 & 7 \end{bmatrix}$.

In option (b) we "share the coefficient equally" between x_1x_2 and x_2x_1, so the matrix B is symmetric. This is the preferred option.
We can do the same for any quadratic form.

Symmetric expressions for quadratic forms

For example, we considered above the quadratic form

$$Q(x) = 10x_1x_2 + 12x_3x_4 - 14x_1x_4 - 16x_2x_3.$$

This can be rewritten symmetrically as

$$Q(x) = 5x_1x_2 + 5x_2x_1 + 6x_3x_4 + 6x_4x_3 - 7x_1x_4 - 7x_4x_1 - 8x_2x_3 - 8x_3x_2,$$

which corresponds to the symmetric matrix

$$B = \begin{bmatrix} 0 & 5 & 0 & -7 \\ 5 & 0 & -8 & 0 \\ 0 & -8 & 0 & 6 \\ -7 & 0 & 6 & 0 \end{bmatrix}$$

$$x^T B x = \begin{bmatrix} x_1 & x_2 & x_3 & x_4 \end{bmatrix} \begin{bmatrix} 0 & 5 & 0 & -7 \\ 5 & 0 & -8 & 0 \\ 0 & -8 & 0 & 6 \\ -7 & 0 & 6 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$$

$$= \begin{bmatrix} x_1 & x_2 & x_3 & x_4 \end{bmatrix} \begin{bmatrix} 5x_2 - 7x_4 \\ 5x_1 - 8x_3 \\ -8x_2 + 6x_4 \\ -7x_1 + 6x_3 \end{bmatrix} = Q(x).$$

Diagonalisation of quadratic forms

Proposition 23.23: Let $Q(x)$ be a quadratic form on $\mathbb{R}^n$.
Then there are integers $r, s \geq 0$ and nonzero vectors $v_1, \dots, v_r, w_1, \dots, w_s$ such that all the v's and w's are orthogonal to each other, and

$$Q(x) = (x.v_1)^2 + \cdots + (x.v_r)^2 - (x.w_1)^2 - \cdots - (x.w_s)^2.$$

Or in terms of linear forms $L_i(x) = x.v_i$ and $M_j(x) = x.w_j$:

$$Q = L_1{}^2 + \cdots + L_r{}^2 - M_1{}^2 - \cdots - M_s{}^2.$$

The *rank* of Q is defined to be $r + s$, and the *signature* is defined to be $r - s$.

Proof: There is a symmetric matrix B such that $Q(x) = x^T Bx$.
By Proposition 23.12, we can find an orthonormal basis $u_1, \dots, u_n$ for $\mathbb{R}^n$ such that each u_i is an eigenvector for B, with eigenvalue $\lambda_i \in \mathbb{R}$ say.
Let r be the number of indices i for which $\lambda_i > 0$,
and let s be the number of indices i for which $\lambda_i < 0$.
We can assume that things have been ordered such that $\lambda_1, \dots, \lambda_r > 0$ and $\lambda_{r+1}, \dots, \lambda_{r+s} < 0$ and any eigenvalues after λ_{r+s} are zero.
Now put $U = [u_1|\cdots|u_n]$ and $D = \mathrm{diag}(\lambda_1, \dots, \lambda_n)$.
We have seen that $B = UDU^T$, so

$$Q(x) = x^T Bx = x^T UDU^T x = (U^T x)^T (DU^T x) = (U^T x).(DU^T x).$$

Diagonalisation of quadratic forms

$$U = [u_1|\cdots|u_n] \qquad D = \mathrm{diag}(\lambda_1, \dots, \lambda_n) \qquad Q(x) = (U^T x).(DU^T x)$$

$$U^T x = \begin{bmatrix} u_1^T \\ \hline \vdots \\ \hline u_n^T \end{bmatrix} x = \begin{bmatrix} u_1.x \\ \vdots \\ u_n.x \end{bmatrix}$$

$$DU^T x = \mathrm{diag}(\lambda_1, \dots, \lambda_n) \begin{bmatrix} u_1.x \\ \vdots \\ u_n.x \end{bmatrix} = \begin{bmatrix} \lambda_1 u_1.x \\ \vdots \\ \lambda_n u_n.x \end{bmatrix}$$

$$Q(x) = (U^T x).(DU^T x) = \begin{bmatrix} u_1.x \\ \vdots \\ u_n.x \end{bmatrix} . \begin{bmatrix} \lambda_1 u_1.x \\ \vdots \\ \lambda_n u_n.x \end{bmatrix} = \lambda_1 (u_1.x)^2 + \cdots + \lambda_n (u_n.x)^2.$$

Diagonalisation of quadratic forms

$$U = [u_1|\cdots|u_n] \qquad D = \mathrm{diag}(\lambda_1, \dots, \lambda_n) \qquad Q(x) = \lambda_1 (u_1.x)^2 + \cdots + \lambda_n (u_n.x)^2.$$

$$\lambda_1, \dots, \lambda_r > 0 \qquad \lambda_{r+1}, \dots, \lambda_{r+s} < 0 \qquad \lambda_{r+s+1}, \dots, \lambda_n = 0$$

- For $1 \leq i \leq r$ we have $\lambda_i > 0$ and we put $v_i = \sqrt{\lambda_i} u_i$ so $\lambda_i (u_i.x)^2 = (v_i.x)^2$.
- For $r + 1 \leq i \leq r + s$ we have $\lambda_i < 0$ and we put $w_{i-r} = \sqrt{|\lambda_i|} u_i$ so $\lambda_i (u_i.x)^2 = -(w_{i-r}.x)^2$.
- For $i > r + s$ we have $\lambda_i = 0$ and $\lambda_i (u_i.x)^2 = 0$.

We thus have

$$Q(x) = (x.v_1)^2 + \cdots + (x.v_r)^2 - (x.w_1)^2 - \cdots - (x.w_s)^2$$

as required.

Example of diagonalising a quadratic form

Consider the quadratic form $Q(x) = x_1 x_2 - x_3 x_4$ on $\mathbb{R}^4$.
It is elementary that for all $a, b \in \mathbb{R}$ we have

$$\left(\frac{a+b}{2}\right)^2 - \left(\frac{a-b}{2}\right)^2 = \frac{a^2 + 2ab + b^2 - a^2 + 2ab - b^2}{4} = ab.$$

Using this, we can rewrite $Q(x)$ as

$$Q(x) = \left(\frac{x_1 + x_2}{2}\right)^2 - \left(\frac{x_1 - x_2}{2}\right)^2 - \left(\frac{x_3 + x_4}{2}\right)^2 + \left(\frac{x_3 - x_4}{2}\right)^2.$$

Now put

$$v_1 = \begin{bmatrix} 1/2 \\ 1/2 \\ 0 \\ 0 \end{bmatrix} \qquad v_2 = \begin{bmatrix} 0 \\ 0 \\ 1/2 \\ -1/2 \end{bmatrix} \qquad w_1 = \begin{bmatrix} 1/2 \\ -1/2 \\ 0 \\ 0 \end{bmatrix} \qquad w_2 = \begin{bmatrix} 0 \\ 0 \\ 1/2 \\ 1/2 \end{bmatrix}.$$

We then have

$$Q(x) = (x.v_1)^2 + (x.v_2)^2 - (x.w_1)^2 - (x.w_2)^2$$

and it is easy to see that the v's and w's are all orthogonal.

Example of diagonalising a quadratic form

Consider the quadratic form $Q(x) = 4x_1x_2 + 6x_2x_3 + 4x_3x_4$ on $\mathbb{R}^4$.
Rewritten symmetrically: $Q(x) = 2x_1x_2 + 2x_2x_1 + 3x_2x_3 + 3x_3x_2 + 2x_3x_4 + 2x_4x_3$.

Corresponding matrix: $B = \begin{bmatrix} 0 & 2 & 0 & 0 \\ 2 & 0 & 3 & 0 \\ 0 & 3 & 0 & 2 \\ 0 & 0 & 2 & 0 \end{bmatrix}$. Characteristic polynomial:

$$\det\begin{bmatrix} -t & 2 & 0 & 0 \\ 2 & -t & 3 & 0 \\ 0 & 3 & -t & 2 \\ 0 & 0 & 2 & -t \end{bmatrix} = -t\det\begin{bmatrix} -t & 3 & 0 \\ 3 & -t & 2 \\ 0 & 2 & -t \end{bmatrix} - 2\det\begin{bmatrix} 2 & 3 & 0 \\ 0 & -t & 2 \\ 0 & 2 & -t \end{bmatrix}$$

$$\det\begin{bmatrix} -t & 3 & 0 \\ 3 & -t & 2 \\ 0 & 2 & -t \end{bmatrix} = -t(t^2-4) - 3(-3t) = 13t - t^3$$

$$\det\begin{bmatrix} 2 & 3 & 0 \\ 0 & -t & 2 \\ 0 & t & -2 \end{bmatrix} = 2(t^2-4) - 3(0-0) = 2t^2 - 8$$

$$\chi_B(t) = -t(13t - t^3) - 2(2t^2 - 8) = t^4 - 17t^2 + 16$$
$$= (t^2-1)(t^2-16) = (t-1)(t+1)(t-4)(t+4).$$

Example of diagonalising a quadratic form

$$Q(x) = x^T Bx \qquad B = \begin{bmatrix} 0 & 2 & 0 & 0 \\ 2 & 0 & 3 & 0 \\ 0 & 3 & 0 & 2 \\ 0 & 0 & 2 & 0 \end{bmatrix} \qquad \chi_B(t) = (t-1)(t+1)(t-4)(t+4)$$

Eigenvalues: $\lambda_1 = 1$, $\lambda_2 = 4$, $\lambda_3 = -1$ and $\lambda_4 = -4$.
Row-reduce $B - \lambda_i I$ to find the eigenvectors:

$$t_1 = \begin{bmatrix} 2 \\ 1 \\ -1 \\ -2 \end{bmatrix} \qquad t_2 = \begin{bmatrix} 1 \\ 2 \\ 2 \\ 1 \end{bmatrix} \qquad t_3 = \begin{bmatrix} 2 \\ -1 \\ -1 \\ 2 \end{bmatrix} \qquad t_4 = \begin{bmatrix} 1 \\ -2 \\ 2 \\ -1 \end{bmatrix}$$

In each case we see that $t_i.t_i = 10$ so the corresponding orthonormal basis consists of the vectors $u_i = t_i/\sqrt{10}$. Following Proposition 23.23:

$$\begin{aligned} v_1 &= \sqrt{\lambda_1}u_1 &&= t_1/\sqrt{10} &&= \sqrt{1/10}\begin{bmatrix} 2 & 1 & -1 & -2 \end{bmatrix}^T \\ v_2 &= \sqrt{\lambda_2}u_2 &&= \sqrt{4}t_2/\sqrt{10} &&= \sqrt{2/5}\begin{bmatrix} 1 & 2 & 2 & 1 \end{bmatrix}^T \\ w_1 &= \sqrt{|\lambda_3|}u_3 &&= t_3/\sqrt{10} &&= \sqrt{1/10}\begin{bmatrix} 2 & -1 & -1 & 2 \end{bmatrix}^T \\ w_2 &= \sqrt{|\lambda_4|}u_4 &&= \sqrt{4}t_4/\sqrt{10} &&= \sqrt{2/5}\begin{bmatrix} 1 & -2 & 2 & -1 \end{bmatrix}^T \end{aligned}$$

Example of diagonalising a quadratic form

$$Q(x) = 4x_1x_2 + 6x_2x_3 + 4x_3x_4$$

$$v_1 = \sqrt{\frac{1}{10}}\begin{bmatrix} 2 \\ 1 \\ -1 \\ -2 \end{bmatrix} \qquad v_2 = \sqrt{\frac{2}{5}}\begin{bmatrix} 1 \\ 2 \\ 2 \\ 1 \end{bmatrix} \qquad w_1 = \sqrt{\frac{1}{10}}\begin{bmatrix} 2 \\ -1 \\ -1 \\ 2 \end{bmatrix} \qquad w_2 = \sqrt{\frac{2}{5}}\begin{bmatrix} 1 \\ -2 \\ 2 \\ -1 \end{bmatrix}$$

Conclusion: $Q(x) = (x.v_1)^2 + (x.v_2)^2 - (x.w_1)^2 - (x.w_2)^2$.