MAS201 PROBLEM SHEET 2

Please hand in your answers for exercises 5 and 11 (together with the exercises mentioned on Problem Sheet 1) in the tutorial in week 3. For the remaining exercises, please enter your answers in the online test system (which can be reached from the course home page at `http://shef.ac.uk/nps/courses/MAS201`). The system will check whether your answers are correct; if not, it will often be able to give feedback about what went wrong.

Lecture 3

Exercise 1. Put

$$p_1 = \begin{bmatrix}1\\2\end{bmatrix} \qquad p_2 = \begin{bmatrix}3\\6\end{bmatrix} \qquad p_3 = \begin{bmatrix}2\\4\end{bmatrix}.$$

Describe geometrically which vectors in $\mathbb{R}^2$ can be expressed as a linear combination of p_1, p_2 and p_3. Give an example of a vector that cannot be described as such a linear combination.

Exercise 2. Put

$$u_1 = \begin{bmatrix}1\\1\\7\end{bmatrix} \qquad u_2 = \begin{bmatrix}2\\2\\3\end{bmatrix} \qquad u_3 = \begin{bmatrix}3\\3\\1\end{bmatrix} \qquad u_4 = \begin{bmatrix}4\\4\\5\end{bmatrix} \qquad u_5 = \begin{bmatrix}5\\5\\2\end{bmatrix}$$

Give an example of a vector $v \in \mathbb{R}^3$ that cannot be expressed as a linear combination of $u_1, \dots, u_5$.

Exercise 3. Consider the vectors

$$a_1 = \begin{bmatrix}1\\1\\1\\1\end{bmatrix} \qquad a_2 = \begin{bmatrix}1\\1\\2\\2\end{bmatrix} \qquad a_3 = \begin{bmatrix}2\\2\\1\\1\end{bmatrix} \qquad a_4 = \begin{bmatrix}1\\2\\2\\1\end{bmatrix} \qquad b = \begin{bmatrix}3\\-2\\0\\5\end{bmatrix}.$$

You may assume the row-reduction

$$\left[\begin{array}{cccc|c} 1 & 1 & 2 & 1 & 3 \\ 1 & 1 & 2 & 2 & -2 \\ 1 & 2 & 1 & 2 & 0 \\ 1 & 2 & 1 & 1 & 5 \end{array}\right] \to \left[\begin{array}{cccc|c} 1 & 0 & 3 & 0 & 6 \\ 0 & 1 & -1 & 0 & 2 \\ 0 & 0 & 0 & 1 & -5 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right].$$

Use this to give a formula expressing b as a linear combination of $a_1, \dots, a_4$.

Exercise 4. Consider the vectors

$$u_1 = \begin{bmatrix}1 & 2 & -1 & 0\end{bmatrix}^T \qquad u_2 = \begin{bmatrix}3 & -1 & 4 & -2\end{bmatrix}^T \qquad u_3 = \begin{bmatrix}-1 & 5 & -6 & 2\end{bmatrix}^T$$

$$v = \begin{bmatrix}5 & -4 & 9 & -4\end{bmatrix}^T \qquad w = \begin{bmatrix}4 & -2 & 3 & 1\end{bmatrix}^T$$

and the matrix

$$A = \left[\begin{array}{c|c|c|c|c} u_1 & u_2 & u_3 & v & w \end{array}\right].$$

(a) Row-reduce A.
(b) Is v a linear combination of u_1, u_2 and u_3?
(c) Is w a linear combination of u_1, u_2 and u_3?

(Note that you do not need any additional row-reductions for parts (b) and (c). Remark 6.7 in the notes is relevant here.)

Exercise 5. Let u_1 and u_2 be vectors in $\mathbb{R}^n$, and put $v_1 = u_1 + u_2$ and $v_2 = u_1 - u_2$.

(a) Show that if a vector w can be expressed as a linear combination of v_1 and v_2, then it can also be expressed as a linear combination of u_1 and u_2.
(b) Give a formula for u_1 in terms of v_1 and v_2, and also a formula for u_2 in terms of v_1 and v_2.
(c) As a converse to (a), show that if a vector w can be expressed as a linear combination of u_1 and u_2, then it can also be expressed as a linear combination of v_1 and v_2.

Exercise 6. Decide whether the following lists are linearly dependent.

(a) $a_1 = \begin{bmatrix} 1 \\ 4 \end{bmatrix}, \quad a_2 = \begin{bmatrix} 5 \\ 3 \end{bmatrix}, \quad a_3 = \begin{bmatrix} 4 \\ 2 \end{bmatrix}, \quad a_4 = \begin{bmatrix} 6 \\ 6 \end{bmatrix}.$

(b) $b_1 = \begin{bmatrix} 5 \\ 0 \\ 0 \\ 3 \end{bmatrix}, \quad b_2 = \begin{bmatrix} 6 \\ 4 \\ 0 \\ 0 \end{bmatrix}, \quad b_3 = \begin{bmatrix} 7 \\ 0 \\ 5 \\ 0 \end{bmatrix}$

(c) $c_1 = \begin{bmatrix} 5 \\ 4 \\ 3 \end{bmatrix}, \quad c_2 = \begin{bmatrix} 4 \\ 5 \\ 4 \end{bmatrix}, \quad c_3 = \begin{bmatrix} 5 \\ 3 \\ 2 \end{bmatrix}$

Exercise 7. Consider the vectors $u = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ and $v = \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}$. Give an example of a nonzero vector w such that the list u, w is independent and the list v, w is independent but the list u, v, w is dependent.

Lecture 4

Exercise 8. Find examples as follows. All your vectors should be nonzero, and all your lists should have length at least two and not contain the same vector twice.

(a) A list of vectors in $\mathbb{R}^3$ that is linearly dependent and does not span $\mathbb{R}^3$.
(b) A list of vectors in $\mathbb{R}^3$ that is linearly dependent and spans $\mathbb{R}^3$.
(c) A list of vectors in $\mathbb{R}^3$ that is linearly independent and does not span $\mathbb{R}^3$.
(d) A list of vectors in $\mathbb{R}^3$ that is linearly independent and spans $\mathbb{R}^3$.

Exercise 9. Decide whether the following statements are true or false. Justify your answers, and give explicit counterexamples for any statements that are false.

(a) Every list of 4 vectors in $\mathbb{R}^3$ spans $\mathbb{R}^3$.
(b) Every list of 4 vectors in $\mathbb{R}^3$ is linearly independent.
(c) If $\mathcal{A}$ is a list that spans $\mathbb{R}^4$ and $\mathcal{B}$ is a linearly independent list in $\mathbb{R}^4$ then $\mathcal{A}$ is at least as long as $\mathcal{B}$.
(d) There is a linearly independent list of length 5 in $\mathbb{R}^6$.

Exercise 10. Consider the list

$$u_1 = \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix}, \quad u_2 = \begin{bmatrix} 0 \\ 3 \\ -2 \end{bmatrix}, \quad u_3 = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}, \quad u_4 = \begin{bmatrix} 3 \\ 0 \\ -1 \end{bmatrix}.$$

Does this span $\mathbb{R}^3$?

Exercise 11. Put $a = \begin{bmatrix} 1 & 3 & 5 & 7 \end{bmatrix} \in \mathbb{R}^4$.

(a) Suppose we have vectors $u_1, \ldots, u_4 \in \mathbb{R}^4$ with $a.u_1 = a.u_2 = a.u_3 = a.u_4 = 0$. Prove that the list $u_1, \ldots, u_4$ does not span $\mathbb{R}^4$.
(b) Give an example of a list $v_1, \ldots, v_4$ that satisfies $a.v_1 = a.v_2 = a.v_3 = a.v_4 = 1$ and also spans $\mathbb{R}^4$.
(c) Give an example of a list $w_1, \ldots, w_4$ that satisfies $a.w_1 = a.w_2 = a.w_3 = a.w_4 = 1$ and does not span $\mathbb{R}^4$.

Exercise 12. The vectors

$$u_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \qquad u_2 = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} \qquad u_3 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \end{bmatrix} \qquad u_4 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \qquad u_5 = \begin{bmatrix} 0 \\ 1 \\ 1 \\ 1 \end{bmatrix} \qquad u_6 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix} \qquad u_7 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

span $\mathbb{R}^4$, because an arbitrary vector $x = \begin{bmatrix} a & b & c & d \end{bmatrix}^T$ can be expressed as a linear combination of u_i by the formula

$$x = (a-b)u_1 + bu_2 + cu_6 + (d-c)u_7,$$

or alternatively by the formula

$$x = -bu_1 + bu_2 - du_3 + (a+d)u_4 - au_5 + cu_6 - cu_7.$$

(a) Check the above formulae.

(b) Give a similar explicit formula to prove that the following vectors span $\mathbb{R}^4$:

$$v_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \end{bmatrix} \qquad v_2 = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 1 \end{bmatrix} \qquad v_3 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \qquad v_4 = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 1 \end{bmatrix} \qquad v_5 = \begin{bmatrix} 0 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

(c) Use the row-reduction method to show again that the vectors v_i span $\mathbb{R}^4$.