MAS201 PROBLEM SHEET 5

Please hand in your answers for exercises 6 and 9 (together with two more exercises that will appear on Problem Sheet 6) in the tutorial in week 8. For the remaining exercises, please enter your answers in the online test system (which can be reached from the course home page at `http://shef.ac.uk/nps/courses/MAS201`).

Lecture 9

Exercise 1. Consider the matrix $A = \begin{bmatrix} 4 & 1 \\ -6 & 9 \end{bmatrix}$. Find an invertible matrix U and a diagonal matrix D such that $A = UDU^{-1}$. Check directly that the equation $A = UDU^{-1}$ holds.

Exercise 2. Show that the matrix $A = \begin{bmatrix} 4 & 1 \\ -1 & 2 \end{bmatrix}$ cannot be diagonalised.

Exercise 3. Consider the matrix

$$A = \begin{bmatrix} 100 & 10 & 1 \\ 100 & 10 & 1 \\ 100 & 10 & 1 \end{bmatrix}.$$

Find a basis for $\mathbb{R}^3$ consisting of eigenvectors for A. Using this, find a diagonalisation $A = UDU^{-1}$.

Exercise 4. Consider the matrix

$$A = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}.$$

Find a basis for $\mathbb{R}^4$ consisting of eigenvectors for A. Using this, find a diagonalisation $A = UDU^{-1}$. Ideally, you should do all this by inspection rather than using the characteristic polynomial and row-reduction.

Exercise 5. Diagonalise the matrix $A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix}$.

Hint: One of the eigenvalues, and the corresponding eigenvector, involves $\sqrt{3}$. You can find another eigenvalue and eigenvector by just changing $\sqrt{3}$ to $-\sqrt{3}$ everywhere. You may also find it useful to remember the rule

$$\frac{1}{a+b\sqrt{3}} = \frac{a-b\sqrt{3}}{(a-b\sqrt{3})(a+b\sqrt{3})} = \frac{a-b\sqrt{3}}{a^2-3b^2}.$$

Lecture 10

Exercise 6. Let A be the 5×5 matrix in which every entry is one.

(a) Show that $A^2 = 5A$.

(b) Suppose that λ is an eigenvalue of A, so there exists a nonzero vector u with $Au = \lambda u$. By considering $A^2 u$, show that $\lambda^2 = 5\lambda$, so $\lambda = 0$ or $\lambda = 5$. (You should not write out any matrices here, or attempt to calculate the characteristic polynomial; just use part (a).)

(c) Find an eigenvector v of eigenvalue 5, and a linearly independent list $w_1, \dots, w_4$ of eigenvectors of eigenvalue 0.

(d) Now put $B = \frac{1}{2}I_5 + \frac{1}{10}A$. Show that B is stochastic.

(e) Prove by induction on k that $B^k = 2^{-k}I_5 + (1-2^{-k})A/5$ for all $k \geq 0$. (You should not write out any matrices here; just use part (a).) What happens when k is large?

Exercise 7. Show that the matrix $A = \begin{bmatrix} 1 & 1 & 0 \\ -1 & 2 & 1 \\ -1 & 0 & 3 \end{bmatrix}$ cannot be diagonalised.
Hint: the eigenvalues are small integers.

Exercise 8. Consider the matrix

$$A = \frac{1}{16} \begin{bmatrix} 10 & 2 & 2 \\ 3 & 11 & 7 \\ 3 & 3 & 7 \end{bmatrix}.$$

For this matrix it turns out that the powers A^n converge to a limit as $n \to \infty$. Use Maple to find a diagonalisation $A = UDU^{-1}$, then find the limit of D^n as $n \to \infty$, then find the limit of A^n.

Exercise 9. Consider the matrix

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}.$$

You may assume that this matrix cannot be diagonalised. Nonetheless, the powers A^n follow a simple pattern. Calculate A^n for some small values of n, then see if you can find the general rule, then prove it by induction.