MAS201 PROBLEM SHEET 6

Please hand in your answers for exercises 3 and 7 (together with the exercises mentioned on Problem Sheet 5) in the tutorial in week 8. For the remaining exercises, please enter your answers in the online test system (which can be reached from the course home page at `http://shef.ac.uk/nps/courses/MAS201`).

Lecture 11

Exercise 1. Solve the following system of differential equations using the method in Section 15:

$$\dot{x} = -2x - 3y \qquad x = 1 \text{ when } t = 0$$
$$\dot{y} = 3x - 2y \qquad y = 0 \text{ when } t = 0.$$

This involves complex eigenvalues. You should remember the rules

$$\cos(t) = (e^{it} + e^{-it})/2 \qquad \sin(t) = (e^{it} - e^{-it})/(2i).$$

Exercise 2. Solve the following system of differential equations:

$$\dot{x}_1 = 0.2x_1 + 0.5x_2 + 0.3x_3$$
$$\dot{x}_2 = 0.6x_1 + 0.6x_2 + 0.7x_3$$
$$\dot{x}_3 = 0.1x_1 + 0.4x_2 + 0.8x_3,$$

with $x = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}^T$ at $t = 0$. You should use Maple to calculate the relevant eigenvalues and eigenvectors. Unlike most examples in this course, this one has not been fine-tuned to work out with nice round numbers.

Exercise 3. Consider the matrix $A = \begin{bmatrix} 1 & 1 \\ 2 & 1 \end{bmatrix}$.

(a) Find the eigenvalues of A.
(b) For each eigenvalue, find a corresponding eigenvector of A.
(c) Define recursively a sequence of vectors $\begin{bmatrix} p_n \\ q_n \end{bmatrix}$ as follows: we have $p_0 = 1$ and $q_0 = 0$, and for all $n > 0$ we have

$$p_n = p_{n-1} + q_{n-1}$$
$$q_n = 2p_{n-1} + q_{n-1}.$$

Use your eigenvectors of A to find expressions for p_n and q_n (for a general positive integer n).

Note: this is not exactly like any of the examples in the notes. You will need to understand the theory behind those examples and adapt it slightly.

Exercise 4. The sequence $(a_n)_{n=0}^{\infty}$ is given by $a_0 = 1001001$, $a_1 = 1010100$, $a_2 = 1110000$ and

$$a_{n+3} = 111a_{n+2} - 1110a_{n+1} + 1000a_n \qquad (\text{ for } n > 2)$$

(a) Write down a matrix equation relating the vector u_n to u_{n+1}, where $u_n = \begin{bmatrix} a_{n+2} \\ a_{n+1} \\ a_n \end{bmatrix}$.
(b) Find the eigenvalues and eigenvectors of the matrix occuring in (a). (If you have done this correctly, the answers will be integers with a nice pattern.)
(c) Express u_0 as a linear combination of the eigenvectors in (b).
(d) Give a general formula for a_n.
Hint: if you do this in the most obvious way, you will need to invert a certain matrix U to give a diagonalisation $A = UDU^{-1}$. However, part (c) provides a shortcut, so you do not need to find U^{-1}.

(e) Check directly that your formula satisfies $a_{n+3} = 111a_{n+2} - 1110a_{n+1} + 1000a_n$ and that a_0, a_1 and a_2 are as they should be.

Exercise 5. Let (a_n) be the sequence given by $a_0 = 2$ and $a_1 = 4$ and $a_{n+2} = 4a_{n+1} - a_n$ for $n \geq 0$. Give a general formula for a_n.
Hint: it is easiest to take a slight shortcut as in step (c) of the previous example.

Lecture 12

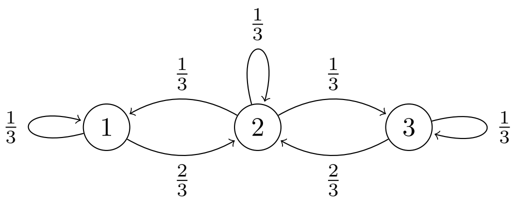
Exercise 6. Over a period of 5 minutes, in a typical MAS201 lecture, 90% of students who are awake at the beginning of the 5-minute period will still be so at the end of it (but the other 10% will fall asleep) and 90% of students who are asleep at the beginning of the 5-minute period will still be so at the end of it (and the other 10% will wake up). If all the students are awake at the beginning of the lecture, what percentage will be awake at the end of the lecture, 50 minutes later?

Exercise 7. Put $d = \begin{bmatrix} 1 & \cdots & 1 \end{bmatrix}^T \in \mathbb{R}^n$.

(a) If $P \in M_n(\mathbb{R})$ is a stochastic matrix, show that $d^T P = d^T$.
(b) Deduce that if $q \in \mathbb{R}^n$ is a probability vector, then Pq is also a probability vector.
(c) Deduce that if $Q \in M_n(\mathbb{R})$ is another stochastic matrix, then PQ is also a stochastic matrix. (**Hint:** how are the columns of PQ related to the columns of Q?)

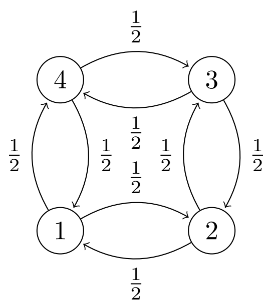
Exercise 8. Suppose that $0 < p, q < 1$, and put $P = \begin{bmatrix} p & 1-q \\ 1-p & q \end{bmatrix}$ (so P is a stochastic matrix). Find the eigenvalues and eigenvectors of P in terms of p and q.
(**Hint:** a general theorem from lectures tells you one of the eigenvalues.)

Exercise 9. Consider the following Markov chain:



Write down the transition matrix and find its eigenvalues and eigenvectors. What is the stationary distribution?

Exercise 10. Consider the following Markov chain:



Write down the transition matrix P and check that $P^3 = P$. Deduce that $P^{2k+1} = P$ for all $k \geq 0$. If we start in state 1 at $t = 0$, what is the probability of being in state 3 at $t = 1111$?
Note: you do not need to calculate any eigenvalues or eigenvectors for this question.