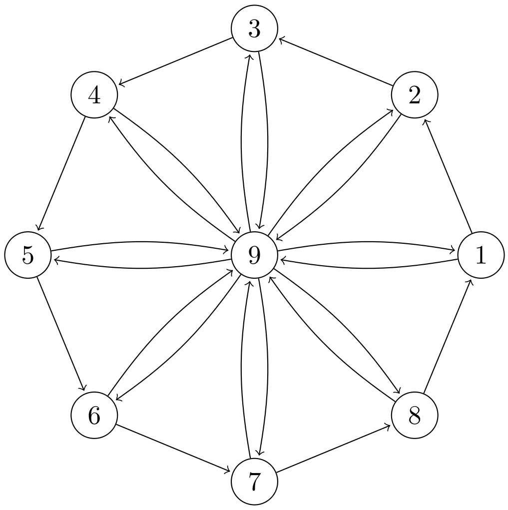
MAS201 PROBLEM SHEET 7

Please hand in your answers for exercises 4 and 5 (together with two more exercises that will appear on Problem Sheet 8) in the tutorial in week 10. For the remaining exercises, please enter your answers in the online test system (which can be reached from the course home page at `http://shef.ac.uk/nps/courses/MAS201`). The online test will let you check your true/false answers for exercise 4, but it cannot check your explanations, which are important.

Lecture 13

Exercise 1. Consider the following web of pages and links.



Let a be the PageRank of page 1, and let b be the PageRank of page 9. By symmetry, pages 2 to 8 must also have rank a. Use the consistency and normalisation conditions to find a and b (without writing down any 9×9 matrices).

Lecture 14

Exercise 2. Each of the following sets is a subspace of $\mathbb{R}^n$ for suitable n.

(a) U_0 is the set of vectors $u = \begin{bmatrix} x & y & z \end{bmatrix}^T$ in $\mathbb{R}^3$ that satisfy $2x - y + 3z = 0$.
(b) U_1 is the set of vectors in $\mathbb{R}^4$ of the form $\begin{bmatrix} s & t-3s & t+2s & t-s \end{bmatrix}^T$.
(c) U_2 is the set of vectors in $\mathbb{R}^4$ that can be expressed as a linear combination of the vectors $a = \begin{bmatrix} 1 & 1 & 1 & 1 \end{bmatrix}^T$ and $b = \begin{bmatrix} 1 & 0 & 0 & 1 \end{bmatrix}^T$.
(d) U_3 is the set of vectors in $\mathbb{R}^3$ that are perpendicular to the vector $c = \begin{bmatrix} 5 & 6 & 7 \end{bmatrix}^T$.

Find two vectors p and q that both lie in U_0. Of course, there are many different answers for this that are equally correct. You should choose your vectors p and q such that they are nonzero and different from each other. Check that $p + q$ is an element of U_0. Then choose examples in the same way for U_1, U_2 and U_3.

Exercise 3. Consider the following sets

$$P_0 = \{\begin{bmatrix} x & y \end{bmatrix}^T \in \mathbb{R}^2 \mid x^2 \geq 1\}$$
$$P_1 = \{\begin{bmatrix} x & y \end{bmatrix}^T \in \mathbb{R}^2 \mid xy \geq 0\}$$
$$P_2 = \{\begin{bmatrix} x & y \end{bmatrix}^T \in \mathbb{R}^2 \mid y \leq x^2\}$$
$$P_3 = \{\begin{bmatrix} x & y \end{bmatrix}^T \in \mathbb{R}^2 \mid x + y \text{ is an integer } \}$$
$$P_4 = \{\begin{bmatrix} x & y \end{bmatrix}^T \in \mathbb{R}^2 \mid x^2 + y^2 \leq 1\}$$

The set P_0 is not closed under addition, because the vectors $u_0 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $u_1 = \begin{bmatrix} -1 \\ 0 \end{bmatrix}$ both lie in P_0, but the sum $u_0 + u_1 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ does not lie in P_0. Moreover, the set P_0 is not closed under scalar multiplication, because the vector $u_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ lies in P_0, but the product $0.5u_2 = \begin{bmatrix} 0.5 \\ 0.5 \end{bmatrix}$ does not lie in P_0. Give similarly specific examples to show that

(a) P_1 is not closed under addition.
(b) P_2 is not closed under addition.
(c) P_2 is not closed under scalar multiplication.
(d) P_3 is not closed under scalar multiplication.
(e) P_4 is not closed under scalar multiplication.

Exercise 4. Which of the following sets is a subspace of $\mathbb{R}^4$?

(a) V_1 is the set of vectors of the form $\begin{bmatrix} s & t+s & t-s & t \end{bmatrix}^T$ (for some $s, t \in \mathbb{R}$).
(b) V_2 is the set of vectors of the form $\begin{bmatrix} t & t^2 & t^3 & t^4 \end{bmatrix}^T$ (for some $t \in \mathbb{R}$).
(c) V_3 is the set of vectors $v = \begin{bmatrix} w & x & y & z \end{bmatrix}^T$ that satisfy $w + 10x + 100y + 1000z = 1$.
(d) V_4 is the set of vectors $v = \begin{bmatrix} w & x & y & z \end{bmatrix}^T$ that satisfy $w - x + y - z = 0$.
(e) V_5 is the set of vectors $v = \begin{bmatrix} w & x & y & z \end{bmatrix}^T$ that satisfy $(w - x)^2 + (y - z)^2 = 0$.

Explain your answers carefully.

Exercise 5. (a) Give an example of a subset $U_0 \subseteq \mathbb{R}^2$ that contains zero and is closed under addition but is not closed under scalar multiplication.
(b) Give an example of a subset $U_1 \subseteq \mathbb{R}^2$ that contains zero and is closed under scalar multiplication but is not closed under addition.
(c) Suppose that U_2 is a nonempty subset of $\mathbb{R}^2$ that is closed under addition and scalar multiplication. Show that U_2 contains the zero vector.
(d) Let U_3 be a subspace of $\mathbb{R}^1 = \mathbb{R}$. Show that U_3 is either $\{0\}$ or all of $\mathbb{R}$.