

MAS201 PROBLEM SHEET 9

Please enter your answers in the online test system (which can be reached from the course home page at <http://shef.ac.uk/nps/courses/MAS201>).

LECTURE 17

Exercise 1. Consider the vectors

$$v_1 = \begin{bmatrix} -1 \\ 2 \\ -1 \\ 3 \end{bmatrix} \quad v_2 = \begin{bmatrix} 1 \\ -1 \\ 2 \\ -2 \end{bmatrix} \quad v_3 = \begin{bmatrix} 1 \\ 0 \\ 3 \\ -1 \end{bmatrix} \quad w_1 = \begin{bmatrix} -1 \\ 5 \\ 2 \\ 6 \end{bmatrix} \quad w_2 = \begin{bmatrix} 1 \\ 1 \\ 4 \\ 0 \end{bmatrix}$$

- Show that $\text{span}(v_1, v_2, v_3) = \text{span}(v_1, v_2) = \text{span}(w_1, w_2)$.
- Find $\dim(\text{span}(v_1, v_2, v_3, w_1, w_2))$.

Exercise 2. Put

$$v_1 = \begin{bmatrix} 1 \\ 3 \\ 5 \\ 3 \end{bmatrix} \quad v_2 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ -3 \end{bmatrix} \quad w_1 = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix} \quad w_2 = \begin{bmatrix} 3 \\ 2 \\ 1 \\ 0 \end{bmatrix}$$

and $V = \text{span}(v_1, v_2)$ and $W = \text{span}(w_1, w_2)$.

- Find the canonical basis for $V + W$.
- Find vectors a_1 and a_2 such that $V = \text{ann}(a_1, a_2)$.
- Find vectors b_1 and b_2 such that $W = \text{ann}(b_1, b_2)$.
- Find the canonical basis for $V \cap W$.

Exercise 3. Put

$$U = \{x \in \mathbb{R}^3 \mid x_1 + 2x_2 + 2x_3 = 0\}$$
$$V = \{x \in \mathbb{R}^3 \mid 4x_1 - x_2 - x_3 = 0\}.$$

Find the canonical bases for U , V , $U + V$ and $U \cap V$.

Exercise 4. Let V be the set of all vectors of the form

$$v = [p + q \quad 2p - 2q \quad 3p + 3q \quad 4p - 4q]^T.$$

- Find vectors v_1 and v_2 such that $V = \text{span}(v_1, v_2)$.
- Find vectors w_1 and w_2 such that $V = \text{ann}(w_1, w_2)$.

Exercise 5. For each of the following configurations, either find an example, or show that no example is possible.

- Subspaces $U, V \leq \mathbb{R}^4$ with $\dim(U) = \dim(V) = 3$ and $\dim(U \cap V) = 1$.
- Subspaces $U, V \leq \mathbb{R}^4$ with $\dim(U) = \dim(V) = 3$ and $\dim(U \cap V) = 2$.
- Subspaces $U, V \leq \mathbb{R}^5$ with $\dim(U) = \dim(V) = 2$ and $\dim(U + V) = 5$.
- Subspaces $U, V \leq \mathbb{R}^3$ with $\dim(U) = \dim(V) = \dim(U + V) = \dim(U \cap V)$.

LECTURE 18

Exercise 6. Find the ranks of the following matrices:

$$A = \begin{bmatrix} 0 & 1 & 2 \\ -1 & 0 & 3 \\ -2 & -3 & 0 \end{bmatrix} \quad B = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 2 & 3 & 4 & 5 & 6 & 7 \\ 3 & 4 & 5 & 6 & 7 & 8 \end{bmatrix} \quad C = \begin{bmatrix} 1 & 10 & 100 \\ 10 & 100 & 1000 \\ 100 & 1000 & 10000 \end{bmatrix} \quad D = \begin{bmatrix} 1 & 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 \end{bmatrix}$$

Exercise 7. Give examples as follows, or explain why no such examples are possible.

- A 3×5 matrix of rank 4.
- A 3×3 matrix of rank 1, in which none of the entries are zero.
- A 2×4 matrix A such that A has rank 1 and A^T has rank 2.
- A 3×3 matrix A such that $A + A^T = 0$ and A has rank 2.
- An invertible 3×3 matrix of rank 2.
- A matrix in RREF with rank 1 and 4 nonzero columns.

Exercise 8. Consider the following matrices, which depend on a parameter t .

$$A = \begin{bmatrix} 1 & 0 \\ 0 & (t-3)(t-4) \end{bmatrix} \quad B = \begin{bmatrix} 1 & t \\ t & 2t-1 \end{bmatrix} \quad C = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & t \\ 1 & 4 & t^2 \end{bmatrix} \quad D = \begin{bmatrix} 1 & 1 & 1 & 1 & 0 \\ 1 & 2 & t & 3 & t \\ 1 & 4 & t^2 & 7 & 3 \end{bmatrix}$$

It should be clear that A usually has rank two, except that when $t = 3$ or $t = 4$ the second row becomes zero and so the rank is only one. In the same way, for each of the other matrices, there is a usual value for the rank, but the rank drops for some exceptional values of t .

- Simplify B by row and column operations. Do not divide any row or column by anything that depends on t , but make B as simple as you can without such divisions.
- What is the usual rank of B ?
- What is the exceptional value of t for which the rank of B is lower? What is the rank in that case?
- What is the usual rank of C , and what are the exceptional cases? (Use the same method as for B .)
- What is the usual rank of D , and what are the exceptional cases? (**Hint:** how is D related to C ?)