MAS201 PROBLEM SHEET 10

This is the last problem sheet. There will not be a corresponding online test.

Exercise 1. Show that if A is an orthogonal matrix then $\det(A) = \pm 1$.

Exercise 2. Consider the matrix

$$A = \frac{1}{25}\begin{bmatrix} 15 & -16 & 12 \\ 20 & 12 & -9 \\ 0 & 15 & 20 \end{bmatrix}.$$

(a) Show that A is an orthogonal matrix.
(b) Check directly that the columns of A form a basis for $\mathbb{R}^3$.
(c) Find the determinant of A.

Exercise 3. Show that the following matrices are both orthogonal.

$$A = \frac{1}{1+t^2}\begin{bmatrix} 1-t^2 & -2t \\ 2t & 1-t^2 \end{bmatrix} \qquad B = \begin{bmatrix} \sin(\phi)\cos(\theta) & \cos(\phi)\cos(\theta) & -\sin(\theta) \\ \sin(\phi)\sin(\theta) & \cos(\phi)\sin(\theta) & \cos(\theta) \\ \cos(\phi) & -\sin(\phi) & 0 \end{bmatrix}$$

Exercise 4. For each of the following, give either an example, or a proof that no example is possible.

(a) An orthonormal list of length 4 in $\mathbb{R}^3$.
(b) A 3×3 matrix that is both symmetric and orthogonal.
(c) A 4×4 matrix that is both antisymmetric and orthogonal.
(d) A 3×3 matrix that is both symmetric and antisymmetric.
(e) A 4×4 orthogonal matrix of rank 3.
(f) A 3×3 matrix that is both antisymmetric and orthogonal (think about determinants).

Exercise 5. There are precisely eight different 3×3 matrices that are both upper triangular and orthogonal. Find them all.

Exercise 6. Find an orthogonal diagonalisation for the following symmetric matrix:

$$A = \begin{bmatrix} 7 & 0 & 4 \\ 0 & -7 & -4 \\ 4 & -4 & 0 \end{bmatrix}.$$

Exercise 7. Consider the matrix

$$A = \begin{bmatrix} 1111 & 1089 & 909 & 891 \\ 1089 & 1111 & 891 & 909 \\ 909 & 891 & 1111 & 1089 \\ 891 & 909 & 1089 & 1111 \end{bmatrix}.$$

Find a diagonalisation $A = UDU^{-1}$, where U is an orthogonal matrix.

Hint: You may assume that there are some eigenvectors of the form $\begin{bmatrix} \pm 1 & \pm 1 & \pm 1 & \pm 1 \end{bmatrix}^T$. As you work through the calculation, you should think carefully at each step about whether the general theory gives you any information that you can use before proceeding further. You should not blindly follow the standard method.

Exercise 8. Diagonalise the quadratic form $Q = ax^2 + 2bxy + ay^2$ (where a and b are constants with $b > a > 0$).

Exercise 9. Diagonalise the quadratic form $Q = 12wx + 10xy + 12yz$, and thus determine the rank and signature.