MAS201 PROBLEM SHEET 1 — Solutions

Lecture 1

Exercise 1. Calculate AB, where

$$A = \begin{bmatrix} 2 & 0 & 0 & 0 \\ 3 & 2 & 0 & 0 \\ 3 & 6 & 2 & 0 \\ 3 & 6 & 3 & 2 \end{bmatrix} \qquad B = \begin{bmatrix} 4 & 2 & 2 & 2 \\ 0 & 1 & -1 & -1 \\ 0 & 0 & 4 & 2 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Solution:

$$AB = \begin{bmatrix} 8 & 4 & 4 & 4 \\ 12 & 8 & 4 & 4 \\ 12 & 12 & 8 & 4 \\ 12 & 12 & 12 & 8. \end{bmatrix}$$

Exercise 2. Consider the following matrices:

$$A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 4 & 3 & 2 & 1 \end{bmatrix} \qquad B = \begin{bmatrix} 1 & 10 \\ 100 & 1000 \end{bmatrix} \qquad C = \begin{bmatrix} 1 & 0 \\ 11 & 0 \\ 111 & 0 \end{bmatrix}.$$

For each of the following products, either evaluate the product or explain why it is undefined:

$$A^2 \qquad AB \qquad AC \qquad BA \qquad B^2 \qquad BC \qquad CA \qquad CB \qquad C^2$$

Solution: The products that are defined are as follows:

$$BA = \begin{bmatrix} 41 & 32 & 23 & 14 \\ 4100 & 3200 & 2300 & 1400 \end{bmatrix}$$

$$B^2 = \begin{bmatrix} 1001 & 10010 \\ 100100 & 1001000 \end{bmatrix}$$

$$CA = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 11 & 22 & 33 & 44 \\ 111 & 222 & 333 & 444 \end{bmatrix}$$

$$CB = \begin{bmatrix} 1 & 10 \\ 11 & 110 \\ 111 & 1110 \end{bmatrix}.$$

The other products are undefined. For example, A is a 2×4 matrix (with 4 columns) and B is a 2×2 matrix (with 2 rows). As the numer of columns in A is different from the number of rows in B, we cannot define the product AB.

Exercise 3. Find examples as follows.

(a) Matrices A and B such that AB is defined but BA is not.
(b) Matrices C and D such that CD and DC are both defined but have different sizes.
(c) Matrices E and F such that EF and FE are both defined and have the same size but are not equal.
(d) Matrices G and H such that GH and HG are both defined and have the same size and are equal.

Solution: In each case there are many possible examples. We will give a selection.

(a) Here A must be an $m \times n$ matrix and B must be an $n \times p$ matrix where m and p are different. We could take $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ (a 2×2 matrix) and $B = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}$ (a 2×3 matrix). The entries in these matrices are not really relevant, only the shape matters. We could therefore simplify things by taking $A = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$. For an even more minimalist example, we could take $A = \begin{bmatrix} 0 \end{bmatrix}$ (a 1×1 matrix) and $B = \begin{bmatrix} 0 & 0 \end{bmatrix}$ (a 1×2 matrix).

(b) Here C must be an $m \times n$ matrix and D must be an $n \times m$ matrix for some integers m and n with $m \neq n$. For a realistic example, we can take

$$C = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 2 & 2 \end{bmatrix} \qquad D = \begin{bmatrix} 3 & 4 \\ 3 & 4 \\ 3 & 4 \end{bmatrix}$$

giving

$$CD = \begin{bmatrix} 9 & 12 \\ 18 & 24 \end{bmatrix} \qquad DC = \begin{bmatrix} 11 & 11 & 11 \\ 11 & 11 & 11 \\ 11 & 11 & 11 \end{bmatrix}.$$

For a minimalist example we can take

$$C = \begin{bmatrix} 0 & 0 \end{bmatrix} \qquad D = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \qquad CD = \begin{bmatrix} 0 \end{bmatrix} \qquad DC = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}.$$

(c) Here E and F must be square matrices of shape $n \times n$ for some $n > 1$. If we choose a pair of 2×2 matrices at random then it will usually work. For example, we could have

$$E = \begin{bmatrix} 1 & 5 \\ 3 & 2 \end{bmatrix} \qquad F = \begin{bmatrix} 3 & 1 \\ 4 & 6 \end{bmatrix} \qquad EF = \begin{bmatrix} 23 & 31 \\ 17 & 15 \end{bmatrix} \qquad FE = \begin{bmatrix} 6 & 17 \\ 22 & 32 \end{bmatrix}.$$

For a minimal example, we have

$$E = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \qquad F = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \qquad EF = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \qquad FE = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}.$$

(d) Here G and H must be square matrices of the same size, say $n \times n$. We can take G to be the zero matrix and H to be any $n \times n$ matrix, and then we have $GH = 0 = HG$, so this gives an example. Alternatively, we can take G to be the identity matrix I_n and H to be any $n \times n$ matrix, and then we have $GH = H = HG$, so this gives another example. Yet another possibility is to let H be any $n \times n$ matrix and then take $G = H$, so that $GH = HG = H^2$. For a minimal example, we can take $n = 1$ and $G = H = \begin{bmatrix} 0 \end{bmatrix}$.

Exercise 4. Find a nonzero matrix A such that A^2 is defined and is zero.

Solution: We could take $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ or $A = \begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix}$.

Exercise 5. The *trace* of a square matrix is the sum of the diagonal entries. Show that if $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ and $B = \begin{bmatrix} p & q \\ r & s \end{bmatrix}$ then the trace of $AB - BA$ is zero.

Solution:

$$\begin{aligned}
AB &= \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} p & q \\ r & s \end{bmatrix} = \begin{bmatrix} ap + br & aq + bs \\ cp + dr & cq + ds \end{bmatrix} \\
BA &= \begin{bmatrix} p & q \\ r & s \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} ap + cq & bp + dq \\ ar + cs & br + ds \end{bmatrix} \\
AB - BA &= \begin{bmatrix} ap + br & aq + bs \\ cp + dr & cq + ds \end{bmatrix} - \begin{bmatrix} ap + cq & bp + dq \\ ar + cs & br + ds \end{bmatrix} \\
&= \begin{bmatrix} br - cq & aq + bs - bp - dq \\ cp + dr - ar - cs & cq - br \end{bmatrix} \\
\text{trace}(AB - BA) &= (br - cq) + (cq - br) = 0.
\end{aligned}$$

Exercise 6. Which of the following matrices are in reduced row-echelon form?

$$A = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad B = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \qquad C = \begin{bmatrix} 1 & 3 & 0 & 2 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$D = \begin{bmatrix} 3 & 1 & 0 & 2 \\ 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \qquad E = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Solution:

- A is not in RREF because the row of zeros occurs at the top, instead of the bottom.
- B is not in RREF because the pivot in the second row is to the left of the pivot in the first row, not to the right.
- C is in RREF.
- D is not in RREF because the first nonzero entry in the first row is equal to 3, not 1. Similarly, the first nonzero entry in the second row is not equal to 1.
- E is not in RREF because the last column contains a nonzero entry above the pivot in the third row.

Exercise 7. Give an example of a 4×7 matrix in RREF with pivots in columns 2, 5 and 7 (and no other columns) and with precisely six nonzero entries.

Solution: Every 4×7 matrix with pivots in the specified columns has the form

$$A = \begin{bmatrix} 0 & 1 & a & b & 0 & c & 0 \\ 0 & 0 & 0 & 0 & 1 & d & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

for some scalars a, b, c and d. If all of these scalars are nonzero then (together with the three pivots) we would have seven nonzero entries in the matrix. We want to have only six nonzero entries, so we can choose $a = b = c = 42$ and $d = 0$ (for example) giving

$$A = \begin{bmatrix} 0 & 1 & 42 & 42 & 0 & 42 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Exercise 8. Use the augmented matrix method to solve the following system of linear equations, or prove that there is no solution.

$$\begin{aligned} 10a &= 10b + c \\ 10c + b &= 10a - 9 \\ a + 100c &= 100b + 11. \end{aligned}$$

Solution: We can tidy up the equations as follows:

$$\begin{array}{rrrl} 10a & -10b & -c & = 0 \\ 10a & -b & -10c & = 9 \\ a & -100b & +100c & = 11. \end{array}$$

Using this we can write down the augmented matrix and row reduce it as follows:

$$\left[\begin{array}{ccc|c} 10 & -10 & -1 & 0 \\ 10 & -1 & -10 & 9 \\ 1 & -100 & 100 & 11 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & -1 & -0.1 & 0 \\ 10 & -1 & -10 & 9 \\ 1 & -100 & 100 & 11 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & -1 & -0.1 & 0 \\ 0 & 9 & -9 & 9 \\ 0 & -99 & 100.1 & 11 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & -1 & -0.1 & 0 \\ 0 & 1 & -1 & 1 \\ 0 & -99 & 100.1 & 11 \end{array}\right]$$

$$\to \left[\begin{array}{ccc|c} 1 & 0 & -1.1 & 1 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & 1.1 & 110 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 0 & -1.1 & 1 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & 1 & 100 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 0 & 0 & 111 \\ 0 & 1 & 0 & 101 \\ 0 & 0 & 1 & 100 \end{array}\right]$$

We conclude that there is a unique solution, namely $a = 111$ and $b = 101$ and $c = 100$.

Exercise 9. Use the augmented matrix method to solve the following system of linear equations, or prove that there is no solution.

$$\begin{aligned} 2w - x - y - 2z &= 1 \\ 3w - 2x - 2y - 3z &= -1 \\ 5w - 3x - 3y - 5z &= 0. \end{aligned}$$

Solution: We can write down the augmented matrix and row-reduce it as follows:

$$\left[\begin{array}{cccc|c} 2 & -1 & -1 & -2 & 1 \\ 3 & -2 & -2 & -3 & -1 \\ 5 & -3 & -3 & -5 & 0 \end{array}\right] \to \left[\begin{array}{cccc|c} 2 & -1 & -1 & -2 & 1 \\ -1 & 0 & 0 & 1 & -3 \\ -1 & 0 & 0 & 1 & -3 \end{array}\right] \to$$

$$\left[\begin{array}{cccc|c} 0 & -1 & -1 & 0 & -5 \\ 1 & 0 & 0 & -1 & 3 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \to \left[\begin{array}{cccc|c} 1 & 0 & 0 & -1 & 3 \\ 0 & 1 & 1 & 0 & 5 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

The final matrix corresponds to the system

$$\begin{aligned} w - z &= 3 \\ x + y &= 5 \\ 0 &= 0. \end{aligned}$$

There are pivots in columns 1 and 2, corresponding to the dependent variables w and x. After rearranging the equations to give the dependent variables in terms of the independent variables, we get $w = z + 3$ and $x = 5 - y$ with y and z arbitrary. Thus, we have an infinite family of solutions.

Exercise 10. Use the augmented matrix method to solve the following system of linear equations, or prove that there is no solution.

$$\begin{aligned} p + q + r &= 30 \\ p + q - r &= 16 \\ p - q + r &= 24 \\ p - q - r &= 11 \end{aligned}$$

Solution: We can write down the augmented matrix and row-reduce it as follows:

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 30 \\ 1 & 1 & -1 & 16 \\ 1 & -1 & 1 & 24 \\ 1 & -1 & -1 & 11 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 1 & 1 & 30 \\ 0 & 0 & -2 & -14 \\ 0 & -2 & 0 & -6 \\ 0 & -2 & -2 & -19 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 1 & 1 & 30 \\ 0 & 0 & 1 & 7 \\ 0 & 1 & 0 & 3 \\ 0 & 1 & 1 & 19/2 \end{array}\right]$$

$$\to \left[\begin{array}{ccc|c} 1 & 0 & 0 & 41/2 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & 7 \\ 0 & 0 & 0 & 2 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right]$$

The final matrix has a pivot in the last column, which means that the original system of equations has no solution.