MAS201 PROBLEM SHEET 2

Lecture 3

Exercise 1. Put

$$p_1 = \begin{bmatrix}1\\2\end{bmatrix} \qquad p_2 = \begin{bmatrix}3\\6\end{bmatrix} \qquad p_3 = \begin{bmatrix}2\\4\end{bmatrix}.$$

Describe geometrically which vectors in $\mathbb{R}^2$ can be expressed as a linear combination of p_1, p_2 and p_3. Give an example of a vector that cannot be described as such a linear combination.

Solution: Any linear combination of the vectors p_i has the form

$$\lambda_1 p_1 + \lambda_2 p_2 + \lambda_3 p_3 = \begin{bmatrix}\lambda_1 + 3\lambda_2 + 2\lambda_3\\2\lambda_1 + 6\lambda_2 + 4\lambda_3\end{bmatrix} = (\lambda_1 + 3\lambda_2 + 2\lambda_3)\begin{bmatrix}1\\2\end{bmatrix}.$$

Thus, these linear combinations are just the multiples of the vector $\begin{bmatrix}1\\2\end{bmatrix}$, so they form the line with equation $y = 2x$. This means that any vector $\begin{bmatrix}x\\y\end{bmatrix}$ with $y \neq 2x$ cannot be expressed as a linear combination of the vectors p_i. For example, the vector $e_1 = \begin{bmatrix}1\\0\end{bmatrix}$ cannot be expressed as a linear combination of the vectors p_i.

Exercise 2. Put

$$u_1 = \begin{bmatrix}1\\1\\7\end{bmatrix} \qquad u_2 = \begin{bmatrix}2\\2\\3\end{bmatrix} \qquad u_3 = \begin{bmatrix}3\\3\\1\end{bmatrix} \qquad u_4 = \begin{bmatrix}4\\4\\5\end{bmatrix} \qquad u_5 = \begin{bmatrix}5\\5\\2\end{bmatrix}$$

Give an example of a vector $v \in \mathbb{R}^3$ that cannot be expressed as a linear combination of $u_1, \dots, u_5$.

Solution: Each of the vectors u_i has the first two components the same, so every linear combination of $u_1, \dots, u_5$ will also have the first two components the same. Thus, if we choose any vector v whose first two components are not the same, then it will not be a linear combination of $u_1, \dots, u_5$. The simplest example is to take $v = e_1 = \begin{bmatrix}1 & 0 & 0\end{bmatrix}^T$.

Exercise 3. Consider the vectors

$$a_1 = \begin{bmatrix}1\\1\\1\\1\end{bmatrix} \qquad a_2 = \begin{bmatrix}1\\1\\2\\2\end{bmatrix} \qquad a_3 = \begin{bmatrix}2\\2\\1\\1\end{bmatrix} \qquad a_4 = \begin{bmatrix}1\\2\\2\\1\end{bmatrix} \qquad b = \begin{bmatrix}3\\-2\\0\\5\end{bmatrix}.$$

You may assume the row-reduction

$$\left[\begin{array}{cccc|c} 1 & 1 & 2 & 1 & 3\\ 1 & 1 & 2 & 2 & -2\\ 1 & 2 & 1 & 2 & 0\\ 1 & 2 & 1 & 1 & 5 \end{array}\right] \to \left[\begin{array}{cccc|c} 1 & 0 & 3 & 0 & 6\\ 0 & 1 & -1 & 0 & 2\\ 0 & 0 & 0 & 1 & -5\\ 0 & 0 & 0 & 0 & 0 \end{array}\right].$$

Use this to give a formula expressing b as a linear combination of $a_1, \dots, a_4$.

Solution: The left hand matrix is $[a_1|a_2|a_3|a_4|b]$, so the row-reduction tells us that the equation $\lambda_1 a_1 + \dots + \lambda_4 a_4 = b$ is equivalent to the system of equations corresponding to the right hand matrix, namely

$$\begin{aligned} \lambda_1 + 3\lambda_3 &= 6\\ \lambda_2 - \lambda_3 &= 2\\ \lambda_4 &= -5. \end{aligned}$$

Here λ_3 is independent so it can take arbitrary values. We can choose $\lambda_3 = 0$, giving $\lambda_1 = 6$ and $\lambda_2 = 2$ and $\lambda_4 = -5$. This means that we have

$$b = \sum_i \lambda_i a_i = 6a_1 + 2a_2 - 5a_4.$$

Exercise 4. Consider the vectors

$$u_1 = \begin{bmatrix} 1 & 2 & -1 & 0 \end{bmatrix}^T \qquad u_2 = \begin{bmatrix} 3 & -1 & 4 & -2 \end{bmatrix}^T \qquad u_3 = \begin{bmatrix} -1 & 5 & -6 & 2 \end{bmatrix}^T$$

$$v = \begin{bmatrix} 5 & -4 & 9 & -4 \end{bmatrix}^T \qquad w = \begin{bmatrix} 4 & -2 & 3 & 1 \end{bmatrix}^T$$

and the matrix

$$A = \left[\begin{array}{c|c|c|c|c} u_1 & u_2 & u_3 & v & w \end{array} \right].$$

(a) Row-reduce A.
(b) Is v a linear combination of u_1, u_2 and u_3?
(c) Is w a linear combination of u_1, u_2 and u_3?

(Note that you do not need any additional row-reductions for parts (b) and (c). Remark 6.7 in the notes is relevant here.)

Solution:

(a) We have

$$A = \left[\begin{array}{c|c|c|c|c} u_1 & u_2 & u_3 & v & w \end{array} \right] = \begin{bmatrix} 1 & 3 & -1 & 5 & 4 \\ 2 & -1 & 5 & -4 & -2 \\ -1 & 4 & -6 & 9 & 3 \\ 0 & -2 & 2 & -4 & 1 \end{bmatrix}$$

$$\to \begin{bmatrix} 1 & 3 & -1 & 5 & 4 \\ 0 & -7 & 7 & -14 & -10 \\ 0 & 7 & -7 & 14 & 7 \\ 0 & -2 & 2 & -4 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 3 & -1 & 5 & 4 \\ 0 & 1 & -1 & 2 & 1 \\ 0 & -7 & 7 & -14 & -10 \\ 0 & -2 & 2 & -4 & 1 \end{bmatrix}$$

$$\to \begin{bmatrix} 1 & 3 & -1 & 5 & 4 \\ 0 & 1 & -1 & 2 & 1 \\ 0 & 0 & 0 & 0 & -3 \\ 0 & 0 & 0 & 0 & 3 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 2 & -1 & 1 \\ 0 & 1 & -1 & 2 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\to \begin{bmatrix} 1 & 0 & 2 & -1 & 0 \\ 0 & 1 & -1 & 2 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

(b) As in Remark 6.7 we can delete the last column and we still have a valid row-reduction

$$\left[\begin{array}{c|c|c|c} u_1 & u_2 & u_3 & v \end{array} \right] = \begin{bmatrix} 1 & 3 & -1 & 5 \\ 2 & -1 & 5 & -4 \\ -1 & 4 & -6 & 9 \\ 0 & -2 & 2 & -4 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 2 & -1 \\ 0 & 1 & -1 & 2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

The matrix on the right is in RREF with no pivot in the last column, which means (by Method 7.6) that v is indeed a linear combination of u_1, u_2 and u_3. More specifically, we see that the equation $\lambda_1 u_1 + \lambda_2 u_2 + \lambda_3 u_3 = v$ is equivalent to the system of equations corresponding to the above matrix, namely

$$\begin{aligned} \lambda_1 + 2\lambda_3 &= -1 \\ \lambda_2 - \lambda_3 &= 2 \\ 0 &= 0 \\ 0 &= 0. \end{aligned}$$

The solution is $\lambda_1 = -1 - 2\lambda_3$ and $\lambda_2 = 2 + \lambda_3$ with λ_3 arbitrary. We can take $\lambda_3 = 0$ giving $\lambda_1 = -1$ and $\lambda_2 = 2$, which means that $v = -u_1 + 2u_2$.

(b) As in Remark 6.7 we can delete the fourth column and we still have a valid row-reduction

$$\left[\begin{array}{c|c|c|c} u_1 & u_2 & u_3 & w \end{array}\right] = \begin{bmatrix} 1 & 3 & -1 & 4 \\ 2 & -1 & 5 & -2 \\ -1 & 4 & -6 & 3 \\ 0 & -2 & 2 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 2 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Here we have a pivot in the last column, indicating that w cannot be expressed as a linear combination of u_1, u_2 and u_3.

Exercise 5. Let u_1 and u_2 be vectors in $\mathbb{R}^n$, and put $v_1 = u_1 + u_2$ and $v_2 = u_1 - u_2$.

(a) Show that if a vector w can be expressed as a linear combination of v_1 and v_2, then it can also be expressed as a linear combination of u_1 and u_2.
(b) Give a formula for u_1 in terms of v_1 and v_2, and also a formula for u_2 in terms of v_1 and v_2.
(c) As a converse to (a), show that if a vector w can be expressed as a linear combination of u_1 and u_2, then it can also be expressed as a linear combination of v_1 and v_2.

Solution:

(a) Suppose that w can be expressed as a linear combination of v_1 and v_2. This means that $w = \lambda_1 v_1 + \lambda_2 v_2$ for some scalars λ_1 and λ_2. After substituting in the definition of v_1 and v_2, we get

$$w = \lambda_1(u_1 + u_2) + \lambda_2(u_1 - u_2) = (\lambda_1 + \lambda_2)u_1 + (\lambda_1 - \lambda_2)u_2.$$

Thus, if we define scalars μ_i by $\mu_1 = \lambda_1 + \lambda_2$ and $\mu_2 = \lambda_1 - \lambda_2$, we have $w = \mu_1 u_1 + \mu_2 u_2$. This expresses w as a linear combination of u_1 and u_2, as required.

(b) By adding the equations $v_1 = u_1 + u_2$ and $v_2 = u_1 - u_2$ we get $2u_1 = v_1 + v_2$ and so $u_1 = v_1/2 + v_2/2$. Similarly, we have $u_2 = v_1/2 - v_2/2$.

(c) Suppose that w can be expressed as a linear combination of u_1 and u_2. This means that $w = \lambda_1 u_1 + \lambda_2 u_2$ for some scalars λ_1 and λ_2. After substituting in the equations from (b) we get

$$w = \lambda_1(v_1/2 + v_2/2) + \lambda_2(v_1/2 - v_2/2) = (\lambda_1/2 + \lambda_2/2)v_1 + (\lambda_1/2 - \lambda_2/2)v_2.$$

Thus, if we define scalars μ_i by $\mu_1 = \lambda_1/2 + \lambda_2/2$ and $\mu_2 = \lambda_1/2 - \lambda_2/2$, we have $w = \mu_1 v_1 + \mu_2 v_2$. This expresses w as a linear combination of v_1 and v_2, as required.

Exercise 6. Decide whether the following lists are linearly dependent.

(a) $a_1 = \begin{bmatrix} 1 \\ 4 \end{bmatrix}, \quad a_2 = \begin{bmatrix} 5 \\ 3 \end{bmatrix}, \quad a_3 = \begin{bmatrix} 4 \\ 2 \end{bmatrix}, \quad a_4 = \begin{bmatrix} 6 \\ 6 \end{bmatrix}.$

(b) $b_1 = \begin{bmatrix} 5 \\ 0 \\ 0 \\ 3 \end{bmatrix}, \quad b_2 = \begin{bmatrix} 6 \\ 4 \\ 0 \\ 0 \end{bmatrix}, \quad b_3 = \begin{bmatrix} 7 \\ 0 \\ 5 \\ 0 \end{bmatrix}$

(c) $c_1 = \begin{bmatrix} 5 \\ 4 \\ 3 \end{bmatrix}, \quad c_2 = \begin{bmatrix} 4 \\ 5 \\ 4 \end{bmatrix}, \quad c_3 = \begin{bmatrix} 5 \\ 3 \\ 2 \end{bmatrix}$

Solution:

(a) Here we have a list of 4 vectors in $\mathbb{R}^2$, and any such list is automatically linearly dependent. (In general, any linearly independent list in $\mathbb{R}^n$ has length at most n, so any list of length greater than n must be dependent.) As an example of a nontrivial linear relation, we have

$$4a_1 + 14a_2 - 8a_3 - 7a_4 = 0.$$

However, we do not need this in order to answer the question as asked.

(b) The list b_1, b_2, b_3 is easily seen to be linearly independent. Indeed, any linear relation $\lambda_1 b_1 + \lambda_2 b_2 + \lambda_3 b_3 = 0$ can be expanded as

$$\begin{bmatrix} 5\lambda_1 + 6\lambda_2 + 7\lambda_3 \\ 4\lambda_2 \\ 5\lambda_3 \\ 3\lambda_1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}.$$

By looking at the fourth entry we see that $3\lambda_1 = 0$ so $\lambda_1 = 0$. Similarly, the second and third entries give $\lambda_2 = \lambda_3 = 0$, so all the λ_i are zero, so our linear relation is the trivial one. As there is only the trivial linear relation, the list is independent.

We can reach the same conclusion by row-reducing the matrix $[b_1|b_2|b_3]$:

$$\begin{bmatrix}5&6&7\\0&4&0\\0&0&5\\3&0&0\end{bmatrix}\to\begin{bmatrix}5&6&7\\0&1&0\\0&0&1\\1&0&0\end{bmatrix}\to\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\\5&6&7\end{bmatrix}\to\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\\0&0&0\end{bmatrix}$$

At the end we have a pivot in every column, so the original list is independent.

(c) Here there is no obvious shortcut so we just row-reduce the matrix $[c_1|c_2|c_3]$:

$$\begin{bmatrix}5&4&5\\4&5&3\\3&4&2\end{bmatrix}\to\begin{bmatrix}5&4&5\\1&1&1\\3&4&2\end{bmatrix}\to\begin{bmatrix}0&-1&0\\1&1&1\\0&1&-1\end{bmatrix}\to$$

$$\begin{bmatrix}0&1&0\\1&0&1\\0&0&-1\end{bmatrix}\to\begin{bmatrix}1&0&1\\0&1&0\\0&0&-1\end{bmatrix}\to\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}$$

Again, we have a pivot in every column, so the list c_1, c_2, c_3 is independent.

Exercise 7. Consider the vectors $u = \begin{bmatrix}1\\2\\3\end{bmatrix}$ and $v = \begin{bmatrix}3\\2\\1\end{bmatrix}$. Give an example of a nonzero vector w such that the list u, w is independent and the list v, w is independent but the list u, v, w is dependent.

Solution: The simplest example is to put $w = u + v = \begin{bmatrix}4\\4\\4\end{bmatrix}$. To see that this works, recall that a list of two nonzero vectors is independent iff the two vectors are not multiples of each other. As w is not a multiple of u, we see that the list u, w is independent. Similarly, as w is not a multiple of v we see that the list v, w is independent. However, we have a nontrivial linear relation $u + v - w = 0$, which proves that the list u, v, w is dependent.

Lecture 4

Exercise 8. Find examples as follows. All your vectors should be nonzero, and all your lists should have length at least two and not contain the same vector twice.

(a) A list of vectors in $\mathbb{R}^3$ that is linearly dependent and does not span $\mathbb{R}^3$.
(b) A list of vectors in $\mathbb{R}^3$ that is linearly dependent and spans $\mathbb{R}^3$.
(c) A list of vectors in $\mathbb{R}^3$ that is linearly independent and does not span $\mathbb{R}^3$.
(d) A list of vectors in $\mathbb{R}^3$ that is linearly independent and spans $\mathbb{R}^3$.

Solution: There are many possible correct solutions. Here is one.

(a) Put $a_1 = e_1 = \begin{bmatrix}1\\0\\0\end{bmatrix}$ and $a_2 = -a_1$. Then the list a_1, a_2 is linearly dependent (because we have a nontrivial linear relation $a_1 + a_2 = 0$) and does not span (because e_2 cannot be written as a linear combination of a_1 and a_2).

(b) Put $b_1 = e_1$ and $b_2 = e_2$ and $b_3 = e_3$ and $b_4 = -e_3$. The list $b_1, \dots, b_4$ is linearly dependent, because we have the nontrivial linear relation $0b_1 + 0b_2 + b_3 + b_4 = 0$. It spans $\mathbb{R}^3$, because any vector $v = \begin{bmatrix}x & y & z\end{bmatrix}^T \in \mathbb{R}^3$ can be written as $v = xb_1 + yb_2 + zb_3 + 0b_4$, which expresses v as a linear combination of $b_1, \dots, b_4$.

(c) Put $c_1 = e_1$ and $c_2 = e_2$. The list c_1, c_2 is clearly linearly independent: a linear relation $\lambda_1 c_1 + \lambda_2 c_2 = 0$ expands to give $\begin{bmatrix}\lambda_1\\\lambda_2\\0\end{bmatrix} = \begin{bmatrix}0\\0\\0\end{bmatrix}$, so $\lambda_1 = \lambda_2 = 0$, so the linear relation is trivial. However, e_3 cannot be expressed as a linear combination of c_1 and c_2, so the list c_1, c_2 does not span.

(d) The list e_1, e_2, e_3 is linearly independent and spans.

Exercise 9. Decide whether the following statements are true or false. Justify your answers, and give explicit counterexamples for any statements that are false.

(a) Every list of 4 vectors in $\mathbb{R}^3$ spans $\mathbb{R}^3$.

(b) Every list of 4 vectors in $\mathbb{R}^3$ is linearly independent.
(c) If $\mathcal{A}$ is a list that spans $\mathbb{R}^4$ and $\mathcal{B}$ is a linearly independent list in $\mathbb{R}^4$ then $\mathcal{A}$ is at least as long as $\mathcal{B}$.
(d) There is a linearly independent list of length 5 in $\mathbb{R}^6$.

Solution:

(a) This is false. For example, the list e_1, e_1, e_1, e_1 is a list of four vectors in $\mathbb{R}^4$ that does not span.
(b) This is also false, and in fact is the opposite of the truth: every list of 4 vectors in $\mathbb{R}^3$ is linearly dependent, not linearly independent.
(c) This is true. As $\mathcal{A}$ spans $\mathbb{R}^4$ it must contain at least 4 vectors, and as $\mathcal{B}$ is linearly independent in $\mathbb{R}^4$ it must contain at most 4 vectors. Thus $\text{length}(\mathcal{B}) \leq 4 \leq \text{length}(\mathcal{A})$.
(d) This is true. The list e_1, e_2, e_3, e_4, e_5 is the most obvious example.

Exercise 10. Consider the list

$$u_1 = \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix}, \quad u_2 = \begin{bmatrix} 0 \\ 3 \\ -2 \end{bmatrix}, \quad u_3 = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}, \quad u_4 = \begin{bmatrix} 3 \\ 0 \\ -1 \end{bmatrix}.$$

Does this span $\mathbb{R}^3$?

Solution: We use Method 9.7, which tells us to perform the following row-reduction:

$$\begin{bmatrix} u_1^T \\ \hline u_2^T \\ \hline u_3^T \\ \hline u_4^T \end{bmatrix} = \begin{bmatrix} 2 & -1 & 0 \\ 0 & 3 & -2 \\ 1 & -2 & 1 \\ 3 & 0 & -1 \end{bmatrix} \to \begin{bmatrix} 0 & 3 & -2 \\ 0 & 3 & -2 \\ 1 & -2 & 1 \\ 0 & 6 & -4 \end{bmatrix} \to$$

$$\begin{bmatrix} 1 & -2 & 1 \\ 0 & 3 & -2 \\ 0 & 3 & -2 \\ 0 & 6 & -4 \end{bmatrix} \to \begin{bmatrix} 1 & -2 & 1 \\ 0 & 1 & -2/3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & -1/3 \\ 0 & 1 & -2/3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

In the final matrix we do not have a pivot in every column, so the specified list does not span $\mathbb{R}^3$.

Exercise 11. Put $a = \begin{bmatrix} 1 & 3 & 5 & 7 \end{bmatrix} \in \mathbb{R}^4$.

(a) Suppose we have vectors $u_1, \ldots, u_4 \in \mathbb{R}^4$ with $a.u_1 = a.u_2 = a.u_3 = a.u_4 = 0$. Prove that the list $u_1, \ldots, u_4$ does not span $\mathbb{R}^4$.
(b) Give an example of a list $v_1, \ldots, v_4$ that satisfies $a.v_1 = a.v_2 = a.v_3 = a.v_4 = 1$ and also spans $\mathbb{R}^4$.
(c) Give an example of a list $w_1, \ldots, w_4$ that satisfies $a.w_1 = a.w_2 = a.w_3 = a.w_4 = 1$ and does not span $\mathbb{R}^4$.

Solution:

(a) If x is a linear combination of the vectors u_i, we have $x = \lambda_1 u_1 + \cdots + \lambda_4 u_4$ for some scalars $\lambda_1, \ldots, \lambda_4$, so

$$a.x = a.(\lambda_1 u_1 + \cdots + \lambda_4 u_4) = \lambda_1(a.u_1) + \cdots + \lambda_4(a.u_4),$$

but $a.u_1 = a.u_2 = a.u_3 = a.u_4 = 0$ so $a.x = 0$. On the other hand, we have $a.e_1 = 1 \neq 0$, so e_1 cannot be a linear combination of the vectors u_i. This means that the u_i do not span $\mathbb{R}^4$.

(b) The obvious example is

$$v_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \qquad v_2 = \begin{bmatrix} 0 \\ 1/3 \\ 0 \\ 0 \end{bmatrix} \qquad v_3 = \begin{bmatrix} 0 \\ 0 \\ 1/5 \\ 0 \end{bmatrix} \qquad v_4 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1/7 \end{bmatrix}.$$

To see that this spans, note that an arbitrary vector $x = \begin{bmatrix} a & b & c & d \end{bmatrix}^T$ in $\mathbb{R}^4$ can be expressed as

$$x = av_1 + 3bv_2 + 5cv_3 + 7dv_4,$$

which is a linear combination of the list $v_1, \ldots, v_4$.

(c) The most obvious solution is to take $w_1 = w_2 = w_3 = w_4 = e_1$. If we prefer to avoid repetitions, we can instead use

$$w_1 = \begin{bmatrix}1\\0\\0\\0\end{bmatrix} \qquad w_2 = \begin{bmatrix}4\\-1\\0\\0\end{bmatrix} \qquad w_3 = \begin{bmatrix}7\\-2\\0\\0\end{bmatrix} \qquad w_4 = \begin{bmatrix}10\\-3\\0\\0\end{bmatrix}.$$

It is clear that any linear combination of $w_1, \dots, w_4$ has zeros in the third and fourth places. In particular, the standard vector e_4 is not a linear combination of the list $w_1, \dots, w_4$, so the list does not span $\mathbb{R}^4$.

Exercise 12. The vectors

$$u_1 = \begin{bmatrix}1\\0\\0\\0\end{bmatrix} \quad u_2 = \begin{bmatrix}1\\1\\0\\0\end{bmatrix} \quad u_3 = \begin{bmatrix}1\\1\\1\\0\end{bmatrix} \quad u_4 = \begin{bmatrix}1\\1\\1\\1\end{bmatrix} \quad u_5 = \begin{bmatrix}0\\1\\1\\1\end{bmatrix} \quad u_6 = \begin{bmatrix}0\\0\\1\\1\end{bmatrix} \quad u_7 = \begin{bmatrix}0\\0\\0\\1\end{bmatrix}$$

span $\mathbb{R}^4$, because an arbitrary vector $x = \begin{bmatrix}a & b & c & d\end{bmatrix}^T$ can be expressed as a linear combination of u_i by the formula

$$x = (a-b)u_1 + bu_2 + cu_6 + (d-c)u_7,$$

or alternatively by the formula

$$x = -bu_1 + bu_2 - du_3 + (a+d)u_4 - au_5 + cu_6 - cu_7.$$

(a) Check the above formulae.
(b) Give a similar explicit formula to prove that the following vectors span $\mathbb{R}^4$:

$$v_1 = \begin{bmatrix}1\\1\\1\\0\end{bmatrix} \quad v_2 = \begin{bmatrix}1\\1\\0\\1\end{bmatrix} \quad v_3 = \begin{bmatrix}1\\1\\1\\1\end{bmatrix} \quad v_4 = \begin{bmatrix}1\\0\\1\\1\end{bmatrix} \quad v_5 = \begin{bmatrix}0\\1\\1\\1\end{bmatrix}$$

(c) Use the row-reduction method to show again that the vectors v_i span $\mathbb{R}^4$.

Solution:

(a) For the first formula we have

$$\begin{aligned}&(a-b)u_1 + bu_2 + cu_6 + (d-c)u_7\\ &= \begin{bmatrix}a-b\\0\\0\\0\end{bmatrix} + \begin{bmatrix}b\\b\\0\\0\end{bmatrix} + \begin{bmatrix}0\\0\\c\\c\end{bmatrix} + \begin{bmatrix}0\\0\\0\\d-c\end{bmatrix} = \begin{bmatrix}a\\b\\c\\d\end{bmatrix}.\end{aligned}$$

For the second, we have

$$\begin{aligned}&-bu_1 + bu_2 - du_3 + (a+d)u_4 - au_5 + cu_6 - cu_7\\ &= \begin{bmatrix}-b\\0\\0\\0\end{bmatrix} + \begin{bmatrix}b\\b\\0\\0\end{bmatrix} + \begin{bmatrix}-d\\-d\\-d\\0\end{bmatrix} + \begin{bmatrix}a+d\\a+d\\a+d\\a+d\end{bmatrix} + \begin{bmatrix}0\\-a\\-a\\-a\end{bmatrix} + \begin{bmatrix}0\\0\\c\\c\end{bmatrix} + \begin{bmatrix}0\\0\\0\\-c\end{bmatrix} = \begin{bmatrix}a\\b\\c\\d\end{bmatrix}.\end{aligned}$$

(b) One possible formula is as follows: if $x = \begin{bmatrix}a & b & c & d\end{bmatrix}^T$, then

$$x = -dv_1 - cv_2 + (a+b+c+d)v_3 - bv_4 - av_5.$$

This can be found as follows: we note that

$$e_1 = v_3 - v_5 \qquad e_2 = v_3 - v_4 \qquad e_3 = v_3 - v_2 \qquad e_4 = v_3 - v_1,$$

and it follows that

$$\begin{aligned}x &= ae_1 + be_2 + ce_3 + de_4\\ &= a(v_3 - v_5) + b(v_3 - v_4) + c(v_3 - v_2) + d(v_3 - v_1)\\ &= -dv_1 - cv_2 + (a+b+c+d)v_3 - bv_4 - av_5.\end{aligned}$$

(c) The general method for these kinds of questions is to construct a matrix A whose rows are the vectors v_i^T, and then row-reduce it:

$$A = \begin{bmatrix} 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & -1 & 0 & 1 \\ 0 & 1 & 1 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 2 \\ 0 & 1 & 1 & 1 \end{bmatrix} \to$$

$$\begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 3 \\ 0 & 1 & 1 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 3 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

The final matrix has a pivot in every column, so the vectors v_i span $\mathbb{R}^4$.