MAS201 PROBLEM SHEET 3

Lecture 5

Exercise 1. You should justify your answers to the following questions.

(a) Is the list $a_1 = \begin{bmatrix} 3 \\ 5 \end{bmatrix}, a_2 = \begin{bmatrix} 2 \\ 7 \end{bmatrix}, a_3 = \begin{bmatrix} 4 \\ 4 \end{bmatrix}$ a basis for $\mathbb{R}^2$?

(b) Is the list $b_1 = \begin{bmatrix} 9 \\ 8 \\ 7 \end{bmatrix}, b_2 = \begin{bmatrix} 8 \\ 7 \\ 6 \end{bmatrix}, b_3 = \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}$ a basis for $\mathbb{R}^3$?

(c) Is the list $c_1 = \begin{bmatrix} 1 \\ 8 \\ 5 \\ 4 \end{bmatrix}, c_2 = \begin{bmatrix} 7 \\ 3 \\ 9 \\ 5 \end{bmatrix}, c_3 = \begin{bmatrix} 5 \\ 1 \\ 9 \\ 9 \end{bmatrix}$ a basis for $\mathbb{R}^4$?

Solution: Any basis for $\mathbb{R}^n$ must contain exactly n vectors. In particular, a basis for $\mathbb{R}^2$ must contain precisely 2 vectors, so a_1, a_2, a_3 cannot be a basis for $\mathbb{R}^2$. (In fact, there is a linear relation $-20a_1 + 8a_2 + 11a_3 = 0$, showing that the list is linearly dependent and so cannot form a basis. However, it is not strictly necessary to work this out.) Similarly, as the list c_1, c_2, c_3 does not have length 4, it cannot form a basis for $\mathbb{R}^4$. This just leaves part (b). Here we can observe that

$$\begin{aligned} b_1 - b_2 &= \begin{bmatrix} 1 & 1 & 1 \end{bmatrix}^T \\ b_2 - b_3 &= \begin{bmatrix} 5 & 5 & 5 \end{bmatrix}^T = 5(b_1 - b_2), \end{aligned}$$

and this rearranges to give a nontrivial linear relation $6b_1 - 5b_2 + b_3 = 0$. This proves that the list b_1, b_2, b_3 is linearly dependent, so again we do not have a basis. This can also be seen by row reducing the matrix $[b_1|b_2|b_3]$:

$$\begin{bmatrix} 9 & 8 & 3 \\ 8 & 7 & 2 \\ 7 & 6 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 8/9 & 1/3 \\ 8 & 7 & 2 \\ 7 & 6 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 8/9 & 1/3 \\ 0 & -1/9 & -2/3 \\ 0 & -2/9 & -4/3 \end{bmatrix} \to \begin{bmatrix} 1 & 8/9 & 1/3 \\ 0 & 1 & 6 \\ 0 & -2/9 & -4/3 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & -5 \\ 0 & 1 & 6 \\ 0 & 0 & 0 \end{bmatrix}$$

As the final result is not the identity matrix, we see that the list b_1, b_2, b_3 is not a basis.

Exercise 2. Consider the list

$$a_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 2 \end{bmatrix}, \quad a_2 = \begin{bmatrix} 1 \\ 1 \\ 3 \\ 4 \end{bmatrix}, \quad a_3 = \begin{bmatrix} 1 \\ 4 \\ 5 \\ 6 \end{bmatrix}, \quad a_4 = \begin{bmatrix} 7 \\ 8 \\ 9 \\ 10 \end{bmatrix}.$$

Is this a basis for $\mathbb{R}^4$?

Solution: We can check this by row-reducing the matrix $[a_1|a_2|a_3|a_4]$:

$$\begin{bmatrix} 1 & 1 & 1 & 7 \\ 1 & 1 & 4 & 8 \\ 1 & 3 & 5 & 9 \\ 2 & 4 & 6 & 10 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & 1 & 7 \\ 0 & 0 & 3 & 1 \\ 0 & 2 & 4 & 2 \\ 0 & 2 & 4 & -4 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & 1 & 7 \\ 0 & 0 & 3 & 1 \\ 0 & 2 & 4 & 2 \\ 0 & 1 & 2 & -2 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & -1 & 9 \\ 0 & 0 & 3 & 1 \\ 0 & 0 & 0 & 6 \\ 0 & 1 & 2 & -2 \end{bmatrix}$$

$$\to \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 2 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

As we end up with the identity matrix, the original list is a basis.

Exercise 3. Put $u_1 = \begin{bmatrix}1\\2\\3\end{bmatrix}$ and $u_2 = \begin{bmatrix}2\\3\\4\end{bmatrix}$. Find a vector u_3 such that the list u_1, u_2, u_3 is a basis for $\mathbb{R}^3$.

Solution: Any vector will do provided that it does not lie in the plane spanned by u_1 and u_2, so if you choose u_3 randomly then it will probably work. The simplest choice is to take $u_3 = e_1 = \begin{bmatrix}1\\0\\0\end{bmatrix}$. To check that u_1, u_2, u_3 is a basis we can row-reduce the matrix $U = [u_1|u_2|u_3]$ and check that we get the identity:

$$\begin{bmatrix}1&2&1\\2&3&0\\3&4&0\end{bmatrix} \to \begin{bmatrix}1&2&1\\0&-1&-2\\0&-2&-3\end{bmatrix} \to \begin{bmatrix}1&0&-3\\0&1&2\\0&0&1\end{bmatrix} \to \begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}.$$

Exercise 4. Suppose that the list a_1, a_2, a_3, a_4, a_5 is a basis for $\mathbb{R}^5$. Show that the list a_1, a_3, a_5 is linearly independent.

Solution: Suppose we have a linear relation $\lambda a_1 + \mu a_3 + \nu a_5 = 0$. This gives a linear relation

$$\lambda a_1 + 0a_2 + \mu a_3 + 0a_4 + \nu a_5 = 0$$

on the whole list. However, the whole list is a basis for $\mathbb{R}^5$, so in particular it is linearly independent. Thus, the above linear relation must be the trivial one, so the coefficients $\lambda, 0, \mu, 0, \nu, 0$ must all be zero. As λ, μ and ν are zero, we see that the original relation on the list a_1, a_3, a_5 is the trivial relation. This means that the list a_1, a_3, a_5 is linearly independent, as claimed.

Lecture 6

Exercise 5. Find the inverse of the matrix

$$A = \begin{bmatrix}0&0&0&1\\0&0&1&1\\0&1&1&1\\1&1&1&1\end{bmatrix}.$$

Solution: We row-reduce the matrix $[A|I_4]$:

$$\left[\begin{array}{cccc|cccc}0&0&0&1&1&0&0&0\\0&0&1&1&0&1&0&0\\0&1&1&1&0&0&1&0\\1&1&1&1&0&0&0&1\end{array}\right] \xrightarrow{1} \left[\begin{array}{cccc|cccc}0&0&0&1&1&0&0&0\\0&0&1&1&0&1&0&0\\0&1&1&1&0&0&1&0\\1&0&0&0&0&0&-1&1\end{array}\right] \xrightarrow{2} \left[\begin{array}{cccc|cccc}0&0&0&1&1&0&0&0\\0&0&1&1&0&1&0&0\\0&1&0&0&0&-1&1&0\\1&0&0&0&0&0&-1&1\end{array}\right]$$

$$\xrightarrow{3} \left[\begin{array}{cccc|cccc}0&0&0&1&1&0&0&0\\0&0&1&0&-1&1&0&0\\0&1&0&0&0&-1&1&0\\1&0&0&0&0&0&-1&1\end{array}\right] \xrightarrow{4} \left[\begin{array}{cccc|cccc}1&0&0&0&0&0&-1&1\\0&1&0&0&0&-1&1&0\\0&0&1&0&-1&1&0&0\\0&0&0&1&1&0&0&0\end{array}\right].$$

The conclusion is that

$$A^{-1} = \begin{bmatrix}0&0&-1&1\\0&-1&1&0\\-1&1&0&0\\1&0&0&0\end{bmatrix}.$$

Exercise 6. Consider the matrix

$$A_0 = \begin{bmatrix}0&10&100&-1&10\\0&11&110&-1&21\\0&-1&-10&0&-11\end{bmatrix}$$

(a) Find a row reduction

$$A_0 \to A_1 \to A_2 \to A_3 \to A_4 \to A_5 \to A_6$$

where each step uses only a single row-operation and A_6 is in RREF.

(b) Find elementary matrices $U_1, \ldots, U_6$ such that $A_i = U_i A_{i-1}$.

(c) Hence find an invertible matrix U such that $A_6 = UA_0$. (Be careful about the order of multiplication.)

Solution: The relevant matrices are as follows:

$$A_1 = \begin{bmatrix} 0 & 10 & 100 & -1 & 10 \\ 0 & 11 & 110 & -1 & 21 \\ 0 & 1 & 10 & 0 & 11 \end{bmatrix} \qquad U_1 = D_3(-1) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

$$A_2 = \begin{bmatrix} 0 & 0 & 0 & -1 & -100 \\ 0 & 11 & 110 & -1 & 21 \\ 0 & 1 & 10 & 0 & 11 \end{bmatrix} \qquad U_2 = E_{13}(-10) = \begin{bmatrix} 1 & 0 & -10 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$A_3 = \begin{bmatrix} 0 & 0 & 0 & -1 & -100 \\ 0 & 0 & 0 & -1 & -100 \\ 0 & 1 & 10 & 0 & 11 \end{bmatrix} \qquad U_3 = E_{23}(-11) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -11 \\ 0 & 0 & 1 \end{bmatrix}$$

$$A_4 = \begin{bmatrix} 0 & 1 & 10 & 0 & 11 \\ 0 & 0 & 0 & -1 & -100 \\ 0 & 0 & 0 & -1 & -100 \end{bmatrix} \qquad U_4 = F_{13} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

$$A_5 = \begin{bmatrix} 0 & 1 & 10 & 0 & 11 \\ 0 & 0 & 0 & 1 & 100 \\ 0 & 0 & 0 & -1 & -100 \end{bmatrix} \qquad U_5 = D_2(-1) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$A_6 = \begin{bmatrix} 0 & 1 & 10 & 0 & 11 \\ 0 & 0 & 0 & 1 & 100 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \qquad U_6 = E_{32}(1) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}.$$

Indeed, the reduction steps are as follows:

(1) Multiply row 3 by -1.
(2) Add -10 times row 3 to row 1.
(3) Add -11 times row 3 ro row 2.
(4) Swap rows 1 and 3.
(5) Multiply row 3 by -1.
(6) Add row 2 to row 3.

The matrices U_i correspond to these row operations as in Proposition 11.3. It follows that

$$\begin{aligned} A_1 &= U_1 A_0 \\ A_2 &= U_2 A_1 = U_2 U_1 A_0 \\ A_3 &= U_3 A_2 = U_3 U_2 U_1 A_0 \end{aligned}$$

and so on, so $A_6 = UA_0$ where $U = U_6 U_5 U_4 U_3 U_2 U_1$. Here

$$U_6 U_5 U_4 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & -1 & 0 \\ 1 & -1 & 0 \end{bmatrix}$$

$$U_3 U_2 U_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -11 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & -10 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 10 \\ 0 & 1 & 11 \\ 0 & 0 & -1 \end{bmatrix}$$

$$U = \begin{bmatrix} 0 & 0 & 1 \\ 0 & -1 & 0 \\ 1 & -1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 10 \\ 0 & 1 & 11 \\ 0 & 0 & -1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & -1 \\ 0 & -1 & -11 \\ 1 & -1 & -1 \end{bmatrix}.$$

As a check, we can verify directly that

$$UA_0 = \begin{bmatrix} 0 & 0 & -1 \\ 0 & -1 & -11 \\ 1 & -1 & -1 \end{bmatrix} \begin{bmatrix} 0 & 10 & 100 & -1 & 10 \\ 0 & 11 & 110 & -1 & 21 \\ 0 & -1 & -10 & 0 & -11 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 10 & 0 & 11 \\ 0 & 0 & 0 & 1 & 100 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} = A_6.$$

Exercise 7. Which of the following matrices are invertible? Justify your answers.

$$A = \begin{bmatrix} 1 & 2 & 2 & 1 \\ 2 & 3 & 3 & 2 \\ 2 & 3 & 3 & 2 \\ 1 & 2 & 2 & 1 \end{bmatrix} \quad B = \begin{bmatrix} 2 & 2 & 2 & 2 \\ 1 & 2 & 2 & 2 \\ 1 & 1 & 2 & 2 \\ 1 & 1 & 1 & 2 \end{bmatrix} \quad C = \begin{bmatrix} 2 & 1 & 1 & 1 \\ 2 & 2 & 1 & 1 \\ 2 & 2 & 2 & 1 \\ 2 & 2 & 2 & 2 \end{bmatrix} \quad D = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 0 & 5 & 6 \\ 0 & 0 & 7 & 8 \\ 0 & 0 & 0 & 9 \end{bmatrix} \quad E = \begin{bmatrix} 1 & 1 \\ 10 & 11 \\ 100 & 111 \\ 1000 & 1111 \end{bmatrix}$$

Solution:

(a) The matrix A is not invertible. Indeed, the first and last rows are the same, as are the middle two rows. Thus, we can perform row operations on A to get a matrix A' with two rows of zeros. It follows that A cannot row-reduce to the identity. Alternatively, we can say that there are only two distinct columns, which means that the columns cannot possibly form a basis for $\mathbb{R}^4$, which again means that the matrix is not invertible.

(b) We can start row-reducing B as follows:

$$B = \begin{bmatrix} 2 & 2 & 2 & 2 \\ 1 & 2 & 2 & 2 \\ 1 & 1 & 2 & 2 \\ 1 & 1 & 1 & 2 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 2 & 2 \\ 1 & 1 & 2 & 2 \\ 1 & 1 & 1 & 2 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix} = B'.$$

As B' is upper-triangular with ones on the diagonal we have $\det(B') = 1$, and it follows that $\det(B) \neq 0$, so B is invertible. More specifically, only the first of our row operations (where we multiplied row 1 by $1/2$) affects the determinant, so $\det(B) = \det(B')/(1/2) = 2$. Alternatively, we can just carry out a few more row operations to see that $B' \to I_4$.

(c) We have $C = B^T$ and it is clear from Theorem 11.5 that the transpose of any invertible matrix is invertible, so C is invertible.

(d) As D is upper triangular, the determinant is the product of the diagonal entries, which is zero because $D_{22} = 0$. It follows that D is not invertible. This can also be seen from the row-reduction

$$\begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 0 & 5 & 6 \\ 0 & 0 & 7 & 8 \\ 0 & 0 & 0 & 9 \end{bmatrix} \to \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 0 & 1 & 6/5 \\ 0 & 0 & 1 & 8/7 \\ 0 & 0 & 0 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 2 & 3 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

(e) The matrix E is not invertible, just because invertibility only makes sense for square matrices.

Exercise 8. Find the inverse of the following matrix, either by creative experimentation or by row-reduction.

$$A = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & a & b \\ 0 & 1 & c & d \end{bmatrix}$$

Solution: The answer is

$$A^{-1} = \begin{bmatrix} -a & -b & 1 & 0 \\ -c & -d & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}.$$

There are various ways to see this. Perhaps the most conceptual is as follows. We can put $B = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ and divide A into 2×2 blocks. We then have $A = \left[\begin{array}{c|c} 0 & I \\ \hline I & B \end{array}\right]$, and the claim is that $A^{-1} = \left[\begin{array}{c|c} -B & I \\ \hline I & 0 \end{array}\right]$. To check this we just need the equation

$$\left[\begin{array}{c|c} 0 & I \\ \hline I & B \end{array}\right] \left[\begin{array}{c|c} -B & I \\ \hline I & 0 \end{array}\right] = \left[\begin{array}{c|c} I & 0 \\ \hline 0 & I \end{array}\right].$$

This is clear provided that we believe that we can treat the 2×2 blocks as though they were just numbers when we perform the above matrix product. This is not completely obvious, but it can be justified.

For a more pedestrian approach, we row-reduce the matrix $[A|I_4]$:

$$\left[\begin{array}{cccc|cccc} 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & a & b & 0 & 0 & 1 & 0 \\ 0 & 1 & c & d & 0 & 0 & 0 & 1 \end{array}\right] \to \left[\begin{array}{cccc|cccc} 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & b & -a & 0 & 1 & 0 \\ 0 & 1 & 0 & d & -c & 0 & 0 & 1 \end{array}\right] \to$$

$$\left[\begin{array}{cccc|cccc} 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & -a & -b & 1 & 0 \\ 0 & 1 & 0 & 0 & -c & -d & 0 & 1 \end{array}\right] \to \left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & -a & -b & 1 & 0 \\ 0 & 1 & 0 & 0 & -c & -d & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 \end{array}\right]$$

(Subtract multiples of row 1 from rows 3 and 4; subtract multiples of row 2 from rows 3 and 4; swap rows 1 and 3, and also swap rows 2 and 4.) The matrix A^{-1} appears as the right hand half of the final result.