MAS201 PROBLEM SHEET 4

Lecture 7

Exercise 1. Calculate the determinant of the matrix

$$A = \begin{bmatrix} a & 0 & b & c \\ d & 0 & 0 & 0 \\ e & f & g & h \\ i & 0 & 0 & j \end{bmatrix}$$

Solution: The most obvious approach is to expand along the top row. This gives

$$\det(A) = a\det(B_1) - 0\det(B_2) + b\det(B_3) + c\det(B_4),$$

where

$$B_1 = \begin{bmatrix} 0 & 0 & 0 \\ f & g & h \\ 0 & 0 & j \end{bmatrix} \qquad B_2 = \begin{bmatrix} d & 0 & 0 \\ e & g & h \\ i & 0 & j \end{bmatrix} \qquad B_3 = \begin{bmatrix} d & 0 & 0 \\ e & f & h \\ i & 0 & j \end{bmatrix} \qquad B_4 = \begin{bmatrix} d & 0 & 0 \\ e & f & g \\ i & 0 & 0 \end{bmatrix}$$

As B_1 has a row of zeros we have $\det(B_1) = 0$. As $\det(B_2)$ gets multiplied by zero, we need not evaluate it. Straightforward expansion gives $\det(B_3) = dfj$ and $\det(B_4) = 0$. Putting this together, we get $\det(A) = bdfj$.

Alternatively, we can expand $\det(A)$ down the second column, and then along the second row, giving

$$\det(A) = (-1)^{3+2} f \det\begin{bmatrix} a & b & c \\ d & 0 & 0 \\ i & 0 & j \end{bmatrix} = (-1)^{3+2}(-1)^{2+1} fd \det\begin{bmatrix} b & c \\ 0 & j \end{bmatrix} = fdbj = bdfj.$$

Exercise 2. Consider the matrix

$$A = \begin{bmatrix} a & b & c & d \\ e & 0 & 0 & f \\ g & 0 & 0 & h \\ i & j & k & l \end{bmatrix}.$$

Prove that $\det(A) = \det\begin{bmatrix} e & f \\ g & h \end{bmatrix} \det\begin{bmatrix} b & c \\ j & k \end{bmatrix}$. (You can reduce the work involved if you choose carefully how to expand the determinant.)

Solution: We expand along the second row. Note that e occurs in the $(2,1)$ position and so comes with a sign $(-1)^{2+1} = -1$, whereas f occurs in the $(2,4)$ position with a sign $(-1)^{2+4} = +1$. We thus have

$$\det(A) = -e\det\begin{bmatrix} b & c & d \\ 0 & 0 & h \\ j & k & l \end{bmatrix} + f\det\begin{bmatrix} a & b & c \\ g & 0 & 0 \\ i & j & k \end{bmatrix}.$$

We now expand out these two 3×3 determinants along the middle row. Note that h is in the $(2,3)$ position of the first 3×3 matrix and so comes with a sign -1, and g is in the $(2,1)$ position of the second 3×3 matrix and so also comes with a sign -1. This gives

$$\det\begin{bmatrix} b & c & d \\ 0 & 0 & h \\ j & k & l \end{bmatrix} = -h\det\begin{bmatrix} b & c \\ j & k \end{bmatrix} = -h(bk - cj)$$

$$\det\begin{bmatrix} a & b & c \\ e & 0 & 0 \\ i & j & k \end{bmatrix} = -g\det\begin{bmatrix} b & c \\ j & k \end{bmatrix} = -g(bk - cj).$$

Putting this together we get

$$\det(A) = (-e)(-h)(bk - cj) + f(-g)(bk - cj) = (eh - fg)(bk - cj) = \det\begin{bmatrix} e & f \\ g & h \end{bmatrix} \det\begin{bmatrix} b & c \\ j & k \end{bmatrix}.$$

Exercise 3. Calculate the determinant of the matrix

$$A = \begin{bmatrix} a & a & a & a \\ a & b & b & b \\ a & b & c & c \\ a & b & c & d \end{bmatrix}.$$

(The easiest method is to start with some carefully chosen row operations as in Method 12.9.)

Solution: We subtract the third row from the fourth row, the second row from the third row, and the first row from the second row to get a new matrix B:

$$A = \begin{bmatrix} a & a & a & a \\ a & b & b & b \\ a & b & c & c \\ a & b & c & d \end{bmatrix} \to \begin{bmatrix} a & a & a & a \\ 0 & b-a & b-a & b-a \\ 0 & 0 & c-b & c-b \\ 0 & 0 & 0 & d-c \end{bmatrix} = B.$$

As we have not swapped any rows or multiplied any rows by a constant, there are no correcting factors and Method 12.9 just tells us that $\det(A) = \det(B)$. As B is upper triangular, the determinant is just the product of the diagonal entries, giving

$$\det(A) = a(b-a)(c-b)(d-c).$$

Exercise 4. Find the adjugate, determinant and inverse of the matrix $C = \begin{bmatrix} a & b & c \\ b & c & a \\ c & a & b \end{bmatrix}$.

(Note that the intermediate calculations that you need for $\det(C)$ are a subset of those that you need for $\operatorname{adj}(C)$. Try not to repeat work unnecessarily.)

Solution: The minors are

$$\begin{aligned}
m_{11} &= \det \begin{bmatrix} c & a \\ a & b \end{bmatrix} = bc - a^2 & m_{12} &= \det \begin{bmatrix} b & a \\ c & b \end{bmatrix} = b^2 - ac & m_{13} &= \det \begin{bmatrix} b & c \\ c & a \end{bmatrix} = ab - c^2 \\
m_{21} &= \det \begin{bmatrix} b & c \\ a & b \end{bmatrix} = b^2 - ac & m_{22} &= \det \begin{bmatrix} a & c \\ c & b \end{bmatrix} = ab - c^2 & m_{23} &= \det \begin{bmatrix} a & b \\ c & a \end{bmatrix} = a^2 - bc \\
m_{31} &= \det \begin{bmatrix} b & c \\ c & a \end{bmatrix} = ab - c^2 & m_{32} &= \det \begin{bmatrix} a & c \\ b & a \end{bmatrix} = a^2 - bc & m_{33} &= \det \begin{bmatrix} a & b \\ b & c \end{bmatrix} = ac - b^2.
\end{aligned}$$

This gives

$$\begin{aligned}
\operatorname{adj}(C) &= \begin{bmatrix} m_{11} & -m_{21} & m_{31} \\ -m_{12} & m_{22} & -m_{32} \\ m_{13} & -m_{23} & m_{33} \end{bmatrix} = \begin{bmatrix} bc-a^2 & ac-b^2 & ab-c^2 \\ ac-b^2 & ab-c^2 & bc-a^2 \\ ab-c^2 & bc-a^2 & ac-b^2 \end{bmatrix} \\
\det(C) &= C_{11}m_{11} - C_{12}m_{12} + C_{13}m_{13} = a(bc-a^2) - b(b^2-ac) + c(ab-c^2) \\
&= 3abc - a^3 - b^3 - c^3 \\
C^{-1} &= \frac{\operatorname{adj}(C)}{\det(C)} = \frac{1}{3abc - a^3 - b^3 - c^3} \begin{bmatrix} bc-a^2 & ac-b^2 & ab-c^2 \\ ac-b^2 & ab-c^2 & bc-a^2 \\ ab-c^2 & bc-a^2 & ac-b^2 \end{bmatrix}
\end{aligned}$$

Exercise 5. Find the adjugate, determinant and inverse of the matrix $H = \begin{bmatrix} 1 & 1/2 & 1/3 \\ 1/2 & 1/3 & 1/4 \\ 1/3 & 1/4 & 1/5 \end{bmatrix}$.

(Note again that the intermediate calculations that you need for $\det(H)$ are a subset of those that you need for $\operatorname{adj}(H)$.)

Solution: The minors are

$$m_{11} = \det\begin{bmatrix}1/3 & 1/4\\ 1/4 & 1/5\end{bmatrix} = \frac{1}{240} \qquad m_{12} = \det\begin{bmatrix}1/2 & 1/4\\ 1/3 & 1/5\end{bmatrix} = \frac{1}{60} \qquad m_{13} = \det\begin{bmatrix}1/2 & 1/3\\ 1/3 & 1/4\end{bmatrix} = \frac{1}{72}$$

$$m_{21} = \det\begin{bmatrix}1/2 & 1/3\\ 1/4 & 1/5\end{bmatrix} = \frac{1}{60} \qquad m_{22} = \det\begin{bmatrix}1 & 1/3\\ 1/3 & 1/5\end{bmatrix} = \frac{4}{45} \qquad m_{23} = \det\begin{bmatrix}1 & 1/2\\ 1/3 & 1/4\end{bmatrix} = \frac{1}{12}$$

$$m_{31} = \det\begin{bmatrix}1/2 & 1/3\\ 1/3 & 1/4\end{bmatrix} = \frac{1}{72} \qquad m_{32} = \det\begin{bmatrix}1 & 1/3\\ 1/2 & 1/4\end{bmatrix} = \frac{1}{12} \qquad m_{33} = \det\begin{bmatrix}1 & 1/2\\ 1/2 & 1/3\end{bmatrix} = \frac{1}{12}.$$

This gives

$$\operatorname{adj}(H) = \begin{bmatrix} m_{11} & -m_{21} & m_{31}\\ -m_{12} & m_{22} & -m_{32}\\ m_{13} & -m_{23} & m_{33}\end{bmatrix} = \begin{bmatrix} \frac{1}{240} & -\frac{1}{60} & \frac{1}{72}\\ -\frac{1}{60} & \frac{4}{45} & -\frac{1}{12}\\ \frac{1}{72} & -\frac{1}{12} & \frac{1}{12}\end{bmatrix}$$

$$\begin{aligned}\det(H) &= H_{11}m_{11} - H_{12}m_{12} + H_{13}m_{13} = \frac{1}{240} - \frac{1}{2}\times\frac{1}{60} + \frac{1}{3}\times\frac{1}{72}\\ &= \frac{1}{2160}\end{aligned}$$

$$\begin{aligned}H^{-1} = \operatorname{adj}(H)/\det(H) &= \begin{bmatrix} \frac{2160}{240} & -\frac{2160}{60} & \frac{2160}{72}\\ -\frac{2160}{60} & \frac{4\times 2160}{45} & -\frac{2160}{12}\\ \frac{2160}{72} & -\frac{2160}{12} & \frac{2160}{12}\end{bmatrix}\\ &= \begin{bmatrix} 9 & -36 & 30\\ -36 & 192 & -180\\ 30 & -180 & 180\end{bmatrix}.\end{aligned}$$

Lecture 8

Exercise 6. Find the characteristic polynomial of the matrix

$$A = \begin{bmatrix} 0 & 0 & 0 & -d\\ 1 & 0 & 0 & -c\\ 0 & 1 & 0 & -b\\ 0 & 0 & 1 & -a\end{bmatrix}$$

Solution: The characteristic polynomial is the determinant of the matrix

$$A - tI = \begin{bmatrix} -t & 0 & 0 & -d\\ 1 & -t & 0 & -c\\ 0 & 1 & -t & -b\\ 0 & 0 & 1 & -a-t\end{bmatrix}.$$

Expanding along the top row, we get

$$\det(A - tI) = -t\det\begin{bmatrix} -t & 0 & -c\\ 1 & -t & -b\\ 0 & 1 & -a-t\end{bmatrix} + d\det\begin{bmatrix} 1 & -t & 0\\ 0 & 1 & -t\\ 0 & 0 & 1\end{bmatrix}.$$

The second matrix above is upper triangular and so the determinant is easily seen to be one. For the first matrix we have

$$\det\begin{bmatrix} -t & 0 & -c\\ 1 & -t & -b\\ 0 & 1 & -a-t\end{bmatrix} = -t\det\begin{bmatrix} -t & -b\\ 1 & -a-t\end{bmatrix} - c\det\begin{bmatrix} 1 & -t\\ 0 & 1\end{bmatrix} = -t(t^2+at+b) - c = -(t^3+at^2+bt+c).$$

Putting this together, we get

$$\det(A - tI) = t(t^3 + at^2 + bt + c) + d = t^4 + at^3 + bt^2 + ct + d.$$

Exercise 7. Find the characteristic polynomial, eigenvalues and all the corresponding eigenvectors of the matrix

$$A = \begin{bmatrix} 1 & 4 & 6 \\ 0 & 2 & 5 \\ 0 & 0 & 3 \end{bmatrix}$$

Solution: The characteristic polynomial is

$$\det(A - tI_3) = \det \begin{bmatrix} 1-t & 4 & 6 \\ 0 & 2-t & 5 \\ 0 & 0 & 3-t \end{bmatrix} = -(t-1)(t-2)(t-3).$$

(Recall that the determinant of an upper triangular 3×3 matrix is the product of the diagonal entries.) Hence the eigenvalues of A are 1, 2 and 3.

To find the eigenvectors of eigenvalue 1, we row-reduce the matrix $A - I$:

$$\begin{bmatrix} 0 & 4 & 6 \\ 0 & 1 & 5 \\ 0 & 0 & 2 \end{bmatrix} \to \begin{bmatrix} 0 & 4 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \to \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}.$$

The final matrix corresponds to the system of equations $y = z = 0$, so the eigenvectors of eigenvalue 1 have the form $\begin{bmatrix} x & 0 & 0 \end{bmatrix}^T$. We take $x = 1$, giving $u_1 = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}^T$ as our first eigenvector.

To find the eigenvectors of eigenvalue 2, we row-reduce the matrix $A - 2I$:

$$\begin{bmatrix} -1 & 4 & 6 \\ 0 & 0 & 5 \\ 0 & 0 & 1 \end{bmatrix} \to \begin{bmatrix} -1 & 4 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & -4 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}.$$

The final matrix corresponds to the system of equations $x - 4y = z = 0$, so the eigenvectors of eigenvalue 1 have the form $\begin{bmatrix} 4y & y & 0 \end{bmatrix}^T$. We take $y = 1$, giving $u_2 = \begin{bmatrix} 4 & 1 & 0 \end{bmatrix}^T$ as our second eigenvector.

To find the eigenvectors of eigenvalue 3, we row-reduce the matrix $A - 3I$:

$$\begin{bmatrix} -2 & 4 & 6 \\ 0 & -1 & 5 \\ 0 & 0 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & -2 & -3 \\ 0 & 1 & -5 \\ 0 & 0 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & -13 \\ 0 & 1 & -5 \\ 0 & 0 & 0 \end{bmatrix}.$$

The final matrix corresponds to the system of equations $x - 13z = y - 5z = 0$, so $x = 13z$ and $y = 5z$. We take $z = 1$, giving $u_3 = \begin{bmatrix} 13 & 5 & 1 \end{bmatrix}^T$ as our third eigenvector.

Exercise 8. Find the characteristic polynomial, eigenvalues and all the corresponding eigenvectors of the matrix

$$B = \begin{bmatrix} 3 & 2 & 1 \\ 0 & 1 & 2 \\ 0 & 1 & -1 \end{bmatrix}$$

Solution: The characteristic polynomial is

$$\det(B - tI) = \det \begin{bmatrix} 3-t & 2 & 1 \\ 0 & 1-t & 2 \\ 0 & 1 & -1-t \end{bmatrix} = -(t-3)((1-t)(-1-t) - 2)$$
$$= -(t-3)(t^2 - 3) = -(t-3)(t-\sqrt{3})(t+\sqrt{3}).$$

Hence the eigenvalues of B are 3, $\sqrt{3}$ and $-\sqrt{3}$.

To find the eigenvectors of eigenvalue 3, we row-reduce the matrix $B - 3I$:

$$\begin{bmatrix} 0 & 2 & 1 \\ 0 & -2 & 2 \\ 0 & 1 & -4 \end{bmatrix} \to \begin{bmatrix} 0 & 0 & 9 \\ 0 & 0 & -6 \\ 0 & 1 & -4 \end{bmatrix} \to \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}. \to \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}.$$

The final matrix corresponds to the system of equations $y = z = 0$, so the eigenvectors of eigenvalue 3 have the form $\begin{bmatrix} x & 0 & 0 \end{bmatrix}^T$. We take $x = 1$, giving $u_1 = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}^T$ as our first eigenvector.

To find the eigenvectors of eigenvalue $\sqrt{3}$, we row-reduce the matrix $B - \sqrt{3}I$:

$$\begin{bmatrix} 3-\sqrt{3} & 2 & 1 \\ 0 & 1-\sqrt{3} & 2 \\ 0 & 1 & -1-\sqrt{3} \end{bmatrix} \xrightarrow{1} \begin{bmatrix} 3-\sqrt{3} & 2 & 1 \\ 0 & 0 & 0 \\ 0 & 1 & -1-\sqrt{3} \end{bmatrix} \xrightarrow{2} \begin{bmatrix} 3-\sqrt{3} & 0 & 3+2\sqrt{3} \\ 0 & 0 & 0 \\ 0 & 1 & -1-\sqrt{3} \end{bmatrix} \xrightarrow{3} \begin{bmatrix} 1 & 0 & (5+3\sqrt{3})/2 \\ 0 & 1 & -1-\sqrt{3} \\ 0 & 0 & 0 \end{bmatrix}.$$

- In step 1 we subtract $1-\sqrt{3}$ times the third row from the second row. The last entry in the second row becomes
$$2-(1-\sqrt{3})(-1-\sqrt{3}) = 2-(-1-\sqrt{3}+\sqrt{3}+\sqrt{3}\sqrt{3}) = 2-2 = 0.$$
- In step 2 we subtract 2 times the third row from the first row. The last entry in the first row becomes
$$1-2(-1-\sqrt{3}) = 3+2\sqrt{3}.$$
- In step 3, we exchange rows two and three, and we also divide the first row by $3-\sqrt{3}$. The last entry in the first row becomes
$$\frac{3+2\sqrt{3}}{3-\sqrt{3}} = \frac{3+2\sqrt{3}}{3-\sqrt{3}}\frac{3+\sqrt{3}}{3+\sqrt{3}} = \frac{15+9\sqrt{3}}{6} = \frac{5+3\sqrt{3}}{2}.$$

The final matrix corresponds to the system of equations
$$\begin{aligned} x+(5+3\sqrt{3})z/2 &= 0 & \qquad x &= -(5+3\sqrt{3})z/2 \\ y-(1+\sqrt{3})z &= 0 & \qquad y &= (1+\sqrt{3})z \end{aligned}$$
with z arbitrary. We choose to take $z=2$, giving
$$u_2 = \begin{bmatrix} -5-3\sqrt{3} & 2+2\sqrt{3} & 2 \end{bmatrix}^T$$
as our second eigenvector.

To find the eigenvectors of eigenvalue $-\sqrt{3}$, we row-reduce the matrix $B+\sqrt{3}I$. The steps are essentially the same as in the previous case, but with all instances of $\sqrt{3}$ changed to $-\sqrt{3}$:
$$\begin{bmatrix} 3+\sqrt{3} & 2 & 1 \\ 0 & 1+\sqrt{3} & 2 \\ 0 & 1 & -1+\sqrt{3} \end{bmatrix} \xrightarrow{1} \begin{bmatrix} 3+\sqrt{3} & 2 & 1 \\ 0 & 0 & 0 \\ 0 & 1 & -1+\sqrt{3} \end{bmatrix} \xrightarrow{2} \begin{bmatrix} 3+\sqrt{3} & 0 & 3-2\sqrt{3} \\ 0 & 0 & 0 \\ 0 & 1 & -1+\sqrt{3} \end{bmatrix} \xrightarrow{3} \begin{bmatrix} 1 & 0 & (5-3\sqrt{3})/2 \\ 0 & 1 & -1+\sqrt{3} \\ 0 & 0 & 0 \end{bmatrix}.$$
This gives
$$u_3 = \begin{bmatrix} -5+3\sqrt{3} & 2-2\sqrt{3} & 2 \end{bmatrix}^T$$
as our third eigenvector.

Exercise 9. Show, directly from the definition of eigenvalue, that 0 is an eigenvalue of the matrix
$$N = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$
Show, also directly from the definition of eigenvalue, that an arbitrary non-zero number k is not an eigenvalue of N. Find all the eigenvectors of N.

Solution: Recall that a k-eigenvector for N is, by definition, a nonzero vector v such that $Nv = kv$.

The 0-eigenvectors for N are the nonzero vectors $v = \begin{bmatrix} w & x & y & z \end{bmatrix}^T$ for which $Nv = 0v = 0$. Now
$$Nv = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} w \\ x \\ y \\ z \end{bmatrix} = \begin{bmatrix} x \\ y \\ z \\ 0 \end{bmatrix},$$
so $Nv = 0$ if and only if $x = y = z = 0$, which means that the 0-eigenvectors are the nonzero vectors of the form $\begin{bmatrix} w & 0 & 0 & 0 \end{bmatrix}^T$, or in other words the nonzero multiples of e_1. In particular, we see that there are some 0-eigenvectors, so 0 is an eigenvalue of N.

Now let k be a nonzero real number. Consider a vector $v = \begin{bmatrix} w & x & y & z \end{bmatrix}^T$ with $Nv = kv$, or equivalently
$$\begin{bmatrix} x \\ y \\ z \\ 0 \end{bmatrix} = \begin{bmatrix} kw \\ kx \\ ky \\ kz \end{bmatrix}.$$
By looking at the last component we see that $kz = 0$, but k is nonzero by assumption, so we must have $z = 0$. We can now look at the third component to see that $ky = z = 0$, and as $k \neq 0$ we can deduce that $y = 0$. Similarly, we can look at the second component to see that $x = 0$, and then look at the first

component to see that $w=0$. This means that v is the zero vector. Thus, there are no nonzero vectors v with $Nv=kv$, which means that k is not an eigenvalue of N. Thus, the only eigenvectors are the 0-eigenvectors, which are the nonzero multiples of e_1, as we worked out previously.

Exercise 10. Find the characteristic polynomial, eigenvalues and eigenvectors of the matrix

$$A=\begin{bmatrix}3&-5&5\\2&-4&5\\2&-2&3\end{bmatrix}.$$

Solution: We need to calculate the characteristic polynomial $\det(A-tI)$. The tidiest way to do this is to simplify $A-tI$ by row operations:

$$\det(A-tI)=\det\begin{bmatrix}3-t&-5&5\\2&-4-t&5\\2&-2&3-t\end{bmatrix}=\det\begin{bmatrix}1-t&t-1&0\\2&-4-t&5\\0&2+t&-2-t\end{bmatrix}=(1-t)(2+t)\det\begin{bmatrix}1&-1&0\\2&-4-t&5\\0&1&-1\end{bmatrix}$$

$$=(1-t)(2+t)\det\begin{bmatrix}1&-1&0\\0&-2-t&5\\0&1&-1\end{bmatrix}=(1-t)(2+t)\det\begin{bmatrix}-2-t&5\\1&-1\end{bmatrix}=(1-t)(2+t)(t-3).$$

(In the first step, we subtract the middle row from the other two rows, which does not change the determinant. In the second step, we extract a factor of $(1-t)$ fom the first row, and a factor of $(2+t)$ from the third row. In the third step, we subtract twice the first row from the second row. Finally, we expand down the first column.)

The more obvious approach is to expand along the first row:

$$\det(A-tI)=(3-t)\det\begin{bmatrix}-4-t&5\\-2&3-t\end{bmatrix}-(-5)\det\begin{bmatrix}2&5\\2&3-t\end{bmatrix}+5\det\begin{bmatrix}2&-4-t\\2&-2\end{bmatrix}$$

$$\det\begin{bmatrix}-4-t&5\\-2&3-t\end{bmatrix}=(-4-t)(3-t)-(-10)=t^2+t-2$$

$$\det\begin{bmatrix}2&5\\2&3-t\end{bmatrix}=(6-2t)-10=-2t-4$$

$$\det\begin{bmatrix}2&-4-t\\2&-2\end{bmatrix}=-4-(-8-2t)=2t+4$$

$$\begin{aligned}\det(A-tI)&=(3-t)(t^2+t-2)+5(-2t-4)+5(2t+4)\\&=(3-t)(t^2+t-2)=(3-t)(t+2)(t-1).\end{aligned}$$

Either way, we see that the eigenvalues are 1, -2 and 3.

To find the eigenvectors of eigenvalue 1, we row-reduce the matrix $A-I$:

$$\begin{bmatrix}2&-5&5\\2&-5&5\\2&-2&2\end{bmatrix}\to\begin{bmatrix}2&-5&5\\0&0&0\\1&-1&1\end{bmatrix}\to\begin{bmatrix}0&-3&3\\0&0&0\\1&-1&1\end{bmatrix}\to\begin{bmatrix}1&-1&1\\0&1&-1\\0&0&0\end{bmatrix}\to\begin{bmatrix}1&0&0\\0&1&-1\\0&0&0\end{bmatrix}.$$

The final matrix corresponds to the equations $x=y-z=0$, so the eigenvectors of eigenvalue 1 have the form $\begin{bmatrix}0&z&z\end{bmatrix}^T$. We take $z=1$, giving $u_1=\begin{bmatrix}0&1&1\end{bmatrix}^T$ as our first eigenvector.

To find the eigenvectors of eigenvalue -2, we row-reduce the matrix $A+2I$:

$$\begin{bmatrix}5&-5&5\\2&-2&5\\2&-2&5\end{bmatrix}\to\begin{bmatrix}1&-1&1\\2&-2&5\\0&0&0\end{bmatrix}\to\begin{bmatrix}1&-1&1\\0&0&3\\0&0&0\end{bmatrix}\to\begin{bmatrix}1&-1&0\\0&0&1\\0&0&0\end{bmatrix}.$$

The final matrix corresponds to the equations $x-y=z=0$, so the eigenvectors of eigenvalue -2 have the form $\begin{bmatrix}y&y&0\end{bmatrix}^T$. We take $y=1$, giving $u_2=\begin{bmatrix}1&1&0\end{bmatrix}^T$ as our second eigenvector.

To find the eigenvectors of eigenvalue 3, we row-reduce the matrix $A-3I$:

$$\begin{bmatrix}0&-5&5\\2&-7&5\\2&-2&0\end{bmatrix}\to\begin{bmatrix}0&1&-1\\2&-7&5\\1&-1&0\end{bmatrix}\to\begin{bmatrix}0&1&-1\\0&-5&5\\1&-1&0\end{bmatrix}\to\begin{bmatrix}0&1&-1\\0&0&0\\1&0&-1\end{bmatrix}\to\begin{bmatrix}1&0&-1\\0&1&-1\\0&0&0\end{bmatrix}.$$

The final matrix corresponds to the equations $x-z=y-z=0$, so the eigenvectors of eigenvalue 3 have the form $\begin{bmatrix}z&z&z\end{bmatrix}^T$. We take $z=1$, giving $u_3=\begin{bmatrix}1&1&1\end{bmatrix}^T$ as our third eigenvector.

Exercise 11. A be an $n \times n$ matrix, and let $\lambda_1, \ldots, \lambda_h$ be h distinct eigenvalues of A. For each $i = 1, \ldots, h$, let the vectors $v_{i,1}, \ldots, v_{i,t_i}$ be linearly independent eigenvectors of A all corresponding to the eigenvalue λ_i. We collect these lists together into a single list

$$v_{1,1}, \ldots, v_{1,t_1}, v_{2,1}, \ldots, v_{2,t_2}, \ldots, v_{h,1}, \ldots, v_{h,t_h}.$$

Prove (as was stated in lectures) that this list is linearly independent.

Solution: For each $i = 1, \ldots, h$, the vectors $v_{i1}, \ldots, v_{it_i}$ are linearly independent eigenvectors of A all corresponding to the eigenvalue λ_i. We show that

$$v_{11}, \ldots, v_{1t_1}, v_{21}, \ldots, v_{2t_2}, \ldots, v_{h1}, \ldots, v_{ht_h}$$

(taken all together) are linearly independent by induction on h.

When $h = 1$, there is nothing to prove, because we are given that $v_{11}, \ldots, v_{1t_1}$ are linearly independent.

Assume now that $h > 1$ and that the claim is true for $h - 1$ distinct eigenvalues of A.

Let

$$a_{11}, \ldots, a_{1t_1}, a_{21}, \ldots, a_{2t_2}, \ldots, a_{h1}, \ldots, a_{ht_h}$$

be scalars such that

$$\sum_{i=1}^{h} \sum_{j=1}^{t_i} a_{ij} v_{ij} = 0. \tag{1}$$

Then

$$0 = A0 = A \left[\sum_{i=1}^{h} \sum_{j=1}^{t_i} a_{ij} v_{ij} \right] = \sum_{i=1}^{h} \sum_{j=1}^{t_i} a_{ij} A v_{ij}$$

and so

$$\sum_{i=1}^{h} \sum_{j=1}^{t_i} a_{ij} \lambda_i v_{ij} = 0 \tag{2}$$

because $Av_{ij} = \lambda_i v_{ij}$ for all $j = 1, \ldots, t_i$ and $i = 1, \ldots, h$. If we now subtract λ_h times (1) from (2) we get

$$\sum_{i=1}^{h} \sum_{j=1}^{t_i} a_{ij} (\lambda_i - \lambda_h) v_{ij} = 0,$$

that is

$$\sum_{i=1}^{h-1} \sum_{j=1}^{t_i} a_{ij} (\lambda_i - \lambda_h) v_{ij} = 0.$$

By the induction hypothesis,

$$v_{11}, \ldots, v_{1t_1}, v_{21}, \ldots, v_{2t_2}, \ldots, v_{h-1,1}, \ldots, v_{h-1,t_{h-1}}$$

(taken all together) are linearly independent. Therefore

$$a_{ij}(\lambda_i - \lambda_h) = 0 \quad \text{for all } j = 1, \ldots, t_i \text{ and } i = 1, \ldots, h-1.$$

Since $\lambda_i - \lambda_h \neq 0$ for all $i = 1, \ldots, h-1$, it follows that

$$a_{ij} = 0 \quad \text{for all } j = 1, \ldots, t_i \text{ and } i = 1, \ldots, h-1.$$

With this information, equation (1) now simplifies to

$$\sum_{j=1}^{t_h} a_{hj} v_{hj} = 0,$$

and so it follows from the fact that $v_{h1}, \ldots, v_{ht_h}$ are linearly independent that $a_{h1} = \cdots = a_{ht_h} = 0$.

We have now shown that

$$v_{11}, \ldots, v_{1t_1}, v_{21}, \ldots, v_{2t_2}, \ldots, v_{h1}, \ldots, v_{ht_h}$$

are linearly independent. This completes the inductive step. By the Principle of Mathematical Induction, the claim is proved.