## MAS201 PROBLEM SHEET 5

### Lecture 9

**Exercise 1.** Consider the matrix $A = \begin{bmatrix} 4 & 1 \\ -6 & 9 \end{bmatrix}$. Find an invertible matrix $U$ and a diagonal matrix $D$ such that $A = UDU^{-1}$. Check directly that the equation $A = UDU^{-1}$ holds.

**Solution:** The characteristic polynomial is

$$\chi_A(t) = \det \begin{bmatrix} 4-t & 1 \\ -6 & 9-t \end{bmatrix} = (4-t)(9-t) - (-6) = t^2 - 13t + 42 = (t-6)(t-7).$$

Thus, the eigenvalues are $\lambda_1 = 6$ and $\lambda_2 = 7$. To find the corresponding eigenvectors we use the following row-reductions:

$$A - \lambda_1 I = \begin{bmatrix} -2 & 1 \\ -6 & 3 \end{bmatrix} \to \begin{bmatrix} 1 & -1/2 \\ -6 & 3 \end{bmatrix} \to \begin{bmatrix} 1 & -1/2 \\ 0 & 0 \end{bmatrix} = B_1$$

$$A - \lambda_2 I = \begin{bmatrix} -3 & 1 \\ -6 & 2 \end{bmatrix} \to \begin{bmatrix} 1 & -1/3 \\ -6 & 2 \end{bmatrix} \to \begin{bmatrix} 1 & -1/3 \\ 0 & 0 \end{bmatrix} = B_2$$

The eigenvector $u_1$ must satisfy $B_1 u_1 = 0$, and it is clear that $u_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ will do. Similarly, the eigenvector $u_2$ must satisfy $B_2 u_2 = 0$, and it is clear that $u_2 = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$ will do. We now take

$$U = [u_1 | u_2] = \begin{bmatrix} 1 & 1 \\ 2 & 3 \end{bmatrix}$$

$$D = \mathrm{diag}(\lambda_1, \lambda_2) = \begin{bmatrix} 6 & 0 \\ 0 & 7 \end{bmatrix}.$$

The general method (Proposition 14.4) tells us that $A = UDU^{-1}$. To check this directly, we need to work out $U^{-1}$. The general formula

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

gives

$$U^{-1} = \frac{1}{3-2} \begin{bmatrix} 3 & -1 \\ -2 & 1 \end{bmatrix} = \begin{bmatrix} 3 & -1 \\ -2 & 1 \end{bmatrix}.$$

We thus have

$$UDU^{-1} = \begin{bmatrix} 1 & 1 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} 6 & 0 \\ 0 & 7 \end{bmatrix} \begin{bmatrix} 3 & -1 \\ -2 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} 18 & -6 \\ -14 & 7 \end{bmatrix} = \begin{bmatrix} 4 & 1 \\ -6 & 9 \end{bmatrix}.$$

As expected, this is the same as $A$.

**Exercise 2.** Show that the matrix $A = \begin{bmatrix} 4 & 1 \\ -1 & 2 \end{bmatrix}$ cannot be diagonalised.

**Solution:** The characteristic polynomial is

$$\det(A - tI) = \det \begin{bmatrix} 4-t & 1 \\ -1 & 2-t \end{bmatrix} = (4-t)(2-t) + 1 = 9 - 6t + t^2 = (t-3)^2.$$

This shows that the only eigenvalue is 3. The eigenvectors of eigenvalue 3 are the vectors $u = \begin{bmatrix} x \\ y \end{bmatrix}$ satisfying $(A - 3I)u = 0$. Here $A - 3I = \begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix}$, so $(A - 3I)u = \begin{bmatrix} x + y \\ -x - y \end{bmatrix}$. This means that $u$ is an eigenvector iff $x + y = 0$, or in other words $u = \begin{bmatrix} x \\ -x \end{bmatrix} = x \begin{bmatrix} 1 \\ -1 \end{bmatrix}$. As every eigenvector is a nonzero multiple of $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$, we see that any two eigenvectors are multiples of each other and so are linearly dependent. Thus, there is no basis of eigenvectors. Proposition 14.4 therefore tells us that $A$ cannot be diagonalised.

**Exercise 3.** Consider the matrix

$$A = \begin{bmatrix} 100 & 10 & 1 \\ 100 & 10 & 1 \\ 100 & 10 & 1 \end{bmatrix}.$$

Find a basis for $\mathbb{R}^3$ consisting of eigenvectors for $A$. Using this, find a diagonalisation $A = UDU^{-1}$.

**Solution:** The characteristic polynomial is as follows.

$$\chi_A(t) = \det \begin{bmatrix} 100-t & 10 & 1 \\ 100 & 10-t & 1 \\ 100 & 10 & 1-t \end{bmatrix} = (100-t)\det \begin{bmatrix} 10-t & 1 \\ 10 & 1-t \end{bmatrix} - 10\det \begin{bmatrix} 100 & 1 \\ 100 & 1-t \end{bmatrix} + \det \begin{bmatrix} 100 & 10-t \\ 100 & 10 \end{bmatrix}$$
$$= (100-t)(t^2 - 11t) - 10(-100t) + (100t) = -t^3 + 111t^2 = -t^2(t-111).$$

It follows that the eigenvalues are 0 and 111. The eigenvectors of eigenvalue 0 are the vectors $u = \begin{bmatrix} x & y & z \end{bmatrix}^T$ satisfying $Au = 0$ or equivalently $100x + 10y + z = 0$. This gives $z = -100x - 10y$, so

$$u = \begin{bmatrix} x \\ y \\ -100x - 10y \end{bmatrix} = x \begin{bmatrix} 1 \\ 0 \\ -100 \end{bmatrix} + y \begin{bmatrix} 0 \\ 1 \\ -10 \end{bmatrix}.$$

Taking $x = 1$ and $y = 0$ gives $u_1 = \begin{bmatrix} 1 & 0 & -100 \end{bmatrix}^T$. Taking $x = 0$ and $y = 1$ gives $u_2 = \begin{bmatrix} 0 \\ 1 \\ -10 \end{bmatrix}$. These are two linearly independent eigenvectors of eigenvalue zero.

Next, to find an eigenvector of eigenvalue 111 we row-reduce the matrix $A - 111I$. If we row-reduce in the obvious way we get the following sequence:

$$\begin{bmatrix} 1 & -10/11 & -1/11 \\ 100 & -101 & 1 \\ 100 & 10 & -110 \end{bmatrix} \to \begin{bmatrix} 1 & -10/11 & -1/11 \\ 0 & -111/11 & 111/11 \\ 100 & 10 & -110 \end{bmatrix} \to \begin{bmatrix} 1 & -10/11 & -1/11 \\ 0 & -111/11 & 111/11 \\ 0 & 1110/11 & -1110/11 \end{bmatrix} \to$$

$$\begin{bmatrix} 1 & -10/11 & -1/11 \\ 0 & 1 & -1 \\ 0 & 1110/11 & -1110/11 \end{bmatrix} \to \begin{bmatrix} 1 & -10/11 & -1/11 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

If we proceed in a more creative order we can avoid fractions:

$$\begin{bmatrix} -11 & 10 & 1 \\ 100 & -101 & 1 \\ 100 & 10 & -110 \end{bmatrix} \to \begin{bmatrix} -11 & 10 & 1 \\ 100 & -101 & 1 \\ 0 & 111 & -111 \end{bmatrix} \to \begin{bmatrix} -11 & 10 & 1 \\ 100 & -101 & 1 \\ 0 & 1 & -1 \end{bmatrix} \to$$

$$\begin{bmatrix} -11 & 0 & 11 \\ 100 & 0 & -100 \\ 0 & 1 & -1 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & -1 \\ 0 & 0 & 0 \\ 0 & 1 & -1 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

Either way, we get the same final matrix $B$. An eigenvector $u = \begin{bmatrix} x & y & z \end{bmatrix}^T$ of eigenvalue 111 must satisfy $Bu = 0$, which means that $x = z$ and $y = z$. Thus, we can take $u_3 = \begin{bmatrix} 1 & 1 & 1 \end{bmatrix}^T$. In fact, if we were sufficiently alert we could have seen that this vector satisfies $Au_3 = 111u_3$ by inspection, and avoided the whole row-reduction process. We now put

$$U = \left[\begin{array}{c|c|c} u_1 & u_2 & u_3 \end{array}\right] = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ -100 & -10 & 1 \end{bmatrix}$$

$$D = \operatorname{diag}(\lambda_1, \lambda_2, \lambda_3) = \operatorname{diag}(0, 0, 111) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 111 \end{bmatrix}.$$

The general theory now tells us that $A = UDU^{-1}$. It would not be hard to check this directly, but the question does not ask us to do so. We just record the value of $U^{-1}$ for any students who wish to check their work:

$$U^{-1} = \frac{1}{111}\begin{bmatrix} 11 & -10 & -1 \\ -100 & 101 & -1 \\ 100 & 10 & 1 \end{bmatrix}.$$

**Exercise 4.** Consider the matrix

$$A = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}.$$

Find a basis for $\mathbb{R}^4$ consisting of eigenvectors for $A$. Using this, find a diagonalisation $A = UDU^{-1}$. Ideally, you should do all this by inspection rather than using the characteristic polynomial and row-reduction.

**Solution:** In terms of the standard basis vectors $e_i$, we have

$$Ae_1 = e_3 \qquad Ae_2 = e_4 \qquad Ae_3 = e_1 \qquad Ae_4 = e_2.$$

It follows that if we put

$$u_1 = e_1 + e_3 \qquad u_2 = e_2 + e_4 \qquad u_3 = e_1 - e_3 \qquad u_4 = e_2 - e_4$$

then

$$Au_1 = u_1 \qquad Au_2 = u_2 \qquad Au_3 = u_3 \qquad Au_4 = u_4,$$

so the vectors $u_i$ are eigenvectors, with eigenvalues $\lambda_1 = \lambda_2 = 1$ and $\lambda_3 = \lambda_4 = -1$. Thus, if we put

$$U = [u_1|u_2|u_3|u_4] = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \end{bmatrix} \qquad D = \operatorname{diag}(\lambda_1, \lambda_2, \lambda_3, \lambda_4) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}$$

then we have $A = UDU^{-1}$. Also, it is not hard to see that $U^2 = 2I_4$, so $U^{-1} = \frac{1}{2}U$.

**Exercise 5.** Diagonalise the matrix $A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix}$.

**Hint:** One of the eigenvalues, and the corresponding eigenvector, involves $\sqrt{3}$. You can find another eigenvalue and eigenvector by just changing $\sqrt{3}$ to $-\sqrt{3}$ everywhere. You may also find it useful to remember the rule

$$\frac{1}{a + b\sqrt{3}} = \frac{a - b\sqrt{3}}{(a - b\sqrt{3})(a + b\sqrt{3})} = \frac{a - b\sqrt{3}}{a^2 - 3b^2}.$$

**Solution:** The characteristic polynomial is

$$\begin{aligned}
\chi_A(t) &= \det \begin{bmatrix} 1-t & 1 & 1 \\ 1 & -t & 1 \\ 1 & 1 & 1-t \end{bmatrix} \\
&= (1-t)\det\begin{bmatrix} -t & 1 \\ 1 & 1-t \end{bmatrix} - \det\begin{bmatrix} 1 & 1 \\ 1 & 1-t \end{bmatrix} + \det\begin{bmatrix} 1 & -t \\ 1 & 1 \end{bmatrix} \\
&= (1-t)(t^2 - t - 1) - (-t) + (1+t) = t^2 - t - 1 - t^3 + t^2 + t + t + 1 + t = -t^3 + 2t^2 + 2t \\
&= -t(t^2 - 2t - 2).
\end{aligned}$$

The quadratic formula tells that the roots of $t^2 - 2t - 2$ are $1 \pm \sqrt{3}$. Thus, the eigenvalues are $\lambda_1 = 0$ and $\lambda_2 = 1 + \sqrt{3}$ and $\lambda_3 = 1 - \sqrt{3}$. By inspection, the vector $u_1 = \begin{bmatrix} 1 & 0 & -1 \end{bmatrix}^T$ satisfies $Au_1 = 0$, so it is an eigenvector of eigenvalue 0. To find an eigenvector of eigenvalue $\lambda_2 = 1 + \sqrt{3}$, we row-reduce the matrix $A - \lambda_2 I$:

$$\begin{bmatrix} -\sqrt{3} & 1 & 1 \\ 1 & -1-\sqrt{3} & 1 \\ 1 & 1 & -\sqrt{3} \end{bmatrix} \xrightarrow{1} \begin{bmatrix} -\sqrt{3} & 1 & 1 \\ 1 & -1-\sqrt{3} & 1 \\ 0 & 2+\sqrt{3} & -1-\sqrt{3} \end{bmatrix} \xrightarrow{2} \begin{bmatrix} 0 & -2-\sqrt{3} & 1+\sqrt{3} \\ 1 & -1-\sqrt{3} & 1 \\ 0 & 2+\sqrt{3} & -1-\sqrt{3} \end{bmatrix} \xrightarrow{3}$$

$$\begin{bmatrix} 0 & 0 & 0 \\ 1 & -1-\sqrt{3} & 1 \\ 0 & 2+\sqrt{3} & -1-\sqrt{3} \end{bmatrix} \xrightarrow{4} \begin{bmatrix} 0 & 0 & 0 \\ 1 & -1-\sqrt{3} & 1 \\ 0 & 1 & 1-\sqrt{3} \end{bmatrix} \xrightarrow{5} \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & -1 \\ 0 & 1 & 1-\sqrt{3} \end{bmatrix} \xrightarrow{6} \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 1-\sqrt{3} \\ 0 & 0 & 0 \end{bmatrix}$$

The steps are as follows:

(1) Subtract row 2 from row 3.
(2) Add $\sqrt{3}$ times row 2 to row 1.
(3) Add row 3 to row 1.
(4) We now want to divide row 3 by $2+\sqrt{3}$. Taking $a = 2$ and $b = 1$ in the equation for $1/(a+b\sqrt{3})$ we get $1/(2+\sqrt{3}) = 2-\sqrt{3}$. We therefore multiply row 3 by $2-\sqrt{3}$.
(5) Add $1+\sqrt{3}$ times row 3 to row 2.
(6) Reorder the rows.

We conclude that an eigenvector $u_2 = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ of eigenvalue $1+\sqrt{3}$ must satisfy $x-z = 0$ and $y+(1-\sqrt{3})z = 0$. Taking $z = 1$ we get $u_2 = \begin{bmatrix} 1 & -1+\sqrt{3} & 1 \end{bmatrix}^T$. Finally, following the hint we see that the final eigenvector $u_3$ is just $\begin{bmatrix} 1 & -1-\sqrt{3} & 1 \end{bmatrix}$ (obtained by changing the $\sqrt{3}$ in $u_2$ to $-\sqrt{3}$). We now have a diagonalisation $A = UDU^{-1}$, where

$$U = \left[ \begin{array}{c|c|c} u_1 & u_2 & u_3 \end{array} \right] = \begin{bmatrix} 1 & 1 & 1 \\ 0 & -1+\sqrt{3} & -1-\sqrt{3} \\ -1 & 1 & 1 \end{bmatrix}$$

$$D = \operatorname{diag}(\lambda_1, \lambda_2, \lambda_3) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1+\sqrt{3} & 0 \\ 0 & 0 & 1-\sqrt{3} \end{bmatrix}.$$

## Lecture 10

**Exercise 6.** Let $A$ be the $5 \times 5$ matrix in which every entry is one.

(a) Show that $A^2 = 5A$.
(b) Suppose that $\lambda$ is an eigenvalue of $A$, so there exists a nonzero vector $u$ with $Au = \lambda u$. By considering $A^2 u$, show that $\lambda^2 = 5\lambda$, so $\lambda = 0$ or $\lambda = 5$. (You should not write out any matrices here, or attempt to calculate the characteristic polynomial; just use part (a).)
(c) Find an eigenvector $v$ of eigenvalue 5, and a linearly independent list $w_1, \dots, w_4$ of eigenvectors of eigenvalue 0.
(d) Now put $B = \frac{1}{2} I_5 + \frac{1}{10} A$. Show that $B$ is stochastic.
(e) Prove by induction on $k$ that $B^k = 2^{-k} I_5 + (1-2^{-k})A/5$ for all $k \geq 0$. (You should not write out any matrices here; just use part (a).) What happens when $k$ is large?

**Solution:**

(a) One way to say this is to introduce the vector $v = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \end{bmatrix}^T$, so $v.v = 5$. We also have

$$A = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \end{bmatrix} = \begin{bmatrix} v^T \\ \hline v^T \\ \hline v^T \\ \hline v^T \\ \hline v^T \end{bmatrix} = \left[ \begin{array}{c|c|c|c|c} v & v & v & v & v \end{array} \right]$$

so

$$A^2 = \begin{bmatrix} v.v & v.v & v.v & v.v & v.v \\ v.v & v.v & v.v & v.v & v.v \\ v.v & v.v & v.v & v.v & v.v \\ v.v & v.v & v.v & v.v & v.v \\ v.v & v.v & v.v & v.v & v.v \end{bmatrix} = \begin{bmatrix} 5 & 5 & 5 & 5 & 5 \\ 5 & 5 & 5 & 5 & 5 \\ 5 & 5 & 5 & 5 & 5 \\ 5 & 5 & 5 & 5 & 5 \\ 5 & 5 & 5 & 5 & 5 \end{bmatrix} = 5A.$$

(b) Suppose we have an eigenvalue $\lambda$, and an associated eigenvector $u$ (so $u \neq 0$ and $Au = \lambda u$). We then have

$$A^2 u = A(Au) = A(\lambda u) = \lambda A u = \lambda^2 u.$$

On the other hand, we have $A^2 = 5A$, so

$$A^2u = 5Au = 5\lambda u.$$

By comparing these two equations, we see that $\lambda^2 u = 5\lambda u$, so $(\lambda^2 - 5\lambda)u = 0$ or $\lambda(\lambda - 5)u = 0$. As $u \neq 0$ it follows that $\lambda = 0$ or $\lambda = 5$.

(c) Put

$$v = \begin{bmatrix}1\\1\\1\\1\\1\end{bmatrix} \qquad w_1 = \begin{bmatrix}1\\-1\\0\\0\\0\end{bmatrix} \qquad w_2 = \begin{bmatrix}0\\1\\-1\\0\\0\end{bmatrix} \qquad w_3 = \begin{bmatrix}0\\0\\1\\-1\\0\end{bmatrix} \qquad w_4 = \begin{bmatrix}0\\0\\0\\1\\-1\end{bmatrix}.$$

It is easy to see that $Av = 5v$ and $Aw_i = 0$ for all $i$, so $v$ is an eigenvector of eigenvalue 5, and $w_1, \dots, w_4$ are eigenvectors of eigenvalue 0. It is also clear that the list $w_1, \dots, w_4$ is linearly independent. This is not the only possible answer. For example, the list

$$w_1' = \begin{bmatrix}1\\0\\0\\0\\-1\end{bmatrix} \qquad w_2' = \begin{bmatrix}0\\1\\0\\0\\-1\end{bmatrix} \qquad w_3' = \begin{bmatrix}0\\0\\1\\0\\-1\end{bmatrix} \qquad w_4' = \begin{bmatrix}0\\0\\0\\1\\-1\end{bmatrix}$$

would do equally well.

(d) In the metrix $B$, every entry away from the diagonal is $\frac{1}{10}$, and every entry on the diagonal is $\frac{1}{10} + \frac{1}{2} = \frac{6}{10}$. In particular, all entries are positive. Moreover, each column contains four entries equal to $\frac{1}{10}$ and one entry equal to $\frac{6}{10}$, adding up to $(4 \times 1 + 6)/10 = 1$. Thus, the matrix is stochastic.

(e) We claim that for all $k \geq 0$ we have $B^k = 2^{-k} I_5 + (1 - 2^{-k})A/5$. When $k = 0$ the left hand side is $B^0 = I_5$, whereas the right hand side is $2^0 I_5 + (1 - 2^0)A = I_5$, as required. When $k = 1$ the left hand side is $B^1 = B = \frac{1}{2} I_5 + \frac{1}{10} A$. We also have $2^{-1} = 1 - 2^{-1} = \frac{1}{2}$ so on the right hand side we have $\frac{1}{2} I_5 + \frac{1}{10} A$ again, as required.

Now suppose that the claim is true for a particular value of $k$. We can multiply the equation $B = \frac{1}{2} I_5 + \frac{1}{10} A$ by the equation $B^k = 2^{-k} I_5 + (1 - 2^{-k})A/5$ and expand out to get

$$\begin{aligned} B^{k+1} &= (\tfrac{1}{2} I_5 + \tfrac{1}{10} A)(2^{-k} I_5 + (1 - 2^{-k})A/5) \\ &= \tfrac{1}{2} 2^{-k} I_5 + \tfrac{1}{2}\tfrac{1}{5}(1 - 2^{-k})A + \tfrac{1}{10} 2^{-k} A + \tfrac{1}{10}\tfrac{1}{5}(1 - 2^{-k})A^2. \end{aligned}$$

Using $A^2 = 5A$ this becomes

$$\begin{aligned} B^{k+1} &= \tfrac{1}{2} 2^{-k} I_5 + \tfrac{1}{2}\tfrac{1}{5}(1 - 2^{-k})A + \tfrac{1}{10} 2^{-k} A + \tfrac{1}{10}(1 - 2^{-k})A \\ &= 2^{-k-1} I_5 + \left(\tfrac{1}{2}(1 - 2^{-k}) + \tfrac{1}{2} 2^{-k} + \tfrac{1}{2}(1 - 2^{-k})\right) A/5 \\ &= 2^{-k-1} I_5 + \left(\tfrac{1}{2} - 2^{-k-1} + 2^{-k-1} + \tfrac{1}{2} - 2^{-k-1})\right) A/5 \\ &= 2^{-k-1} I_5 + (1 - 2^{-k-1})A/5. \end{aligned}$$

This is the case $k + 1$ of our claim. It follows by induction that the claim holds for all $k$.

**Exercise 7.** Show that the matrix $A = \begin{bmatrix}1 & 1 & 0\\-1 & 2 & 1\\-1 & 0 & 3\end{bmatrix}$ cannot be diagonalised.

**Hint:** the eigenvalues are small integers.

**Solution:** The characteristic polynomial is

$$\begin{aligned} \chi_A(t) &= \det \begin{bmatrix}1-t & 1 & 0\\-1 & 2-t & 1\\-1 & 0 & 3-t\end{bmatrix} = (1-t)\det\begin{bmatrix}2-t & 1\\0 & 3-t\end{bmatrix} - \det\begin{bmatrix}-1 & 1\\-1 & 3-t\end{bmatrix} \\ &= (1-t)(2-t)(3-t) - (t-2) = (2-t)((1-t)(3-t)+1) \\ &= (2-t)(4-4t+t^2) = (2-t)^3. \end{aligned}$$

(If we had not spotted that $2 - t$ was a common factor and had just expanded everything out, we would have found that $\chi_A(t) = -t^3 + 6t^2 - 12t + 8$. Using the hint we could have tried various small integers and found that $\chi_A(2) = 0$, then we could have divided $\chi_A(t)$ by $t - 2$ to get $-t^2 + 4t - 4$, then we could have used the quadratic formula to see that 2 is the only root.)

We now see that 2 is the only eigenvalue of $A$. To find the eigenvectors, we row-reduce $A - 2I$:

$$\begin{bmatrix} -1 & 1 & 0 \\ -1 & 0 & 1 \\ -1 & 0 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & -1 & 0 \\ 0 & -1 & 1 \\ 0 & -1 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

From this we see that the eigenvectors of eigenvalue 2 are just the nonzero vectors of the form $u = \begin{bmatrix} x & x & x \end{bmatrix}^T$. In particular, any two eigenvectors are multiples of each other, and so are linearly dependent. It follows that there is no basis of eigenvectors, so the matrix cannot be diagonalised.

**Exercise 8.** Consider the matrix

$$A = \frac{1}{16} \begin{bmatrix} 10 & 2 & 2 \\ 3 & 11 & 7 \\ 3 & 3 & 7 \end{bmatrix}.$$

For this matrix it turns out that the powers $A^n$ converge to a limit as $n \to \infty$. Use Maple to find a diagonalisation $A = UDU^{-1}$, then find the limit of $D^n$ as $n \to \infty$, then find the limit of $A^n$.

**Solution:** We enter the definition of $A$ and find the eigenvectors as follows:

```
with(LinearAlgebra):

A := <<10|2|2>,<3|11|7>,<3|3|7>>/16;

L,U := Eigenvectors(A);
```

Maple responds by printing

$$L, U := \begin{bmatrix} 1 \\ 1/4 \\ 1/2 \end{bmatrix}, \begin{bmatrix} 1 & 0 & -1 \\ 2 & -1 & 1 \\ 1 & 1 & 0 \end{bmatrix}$$

This indicates that the eigenvalues are $\lambda_1 = 1$, $\lambda_2 = \frac{1}{4}$ and $\lambda_3 = \frac{1}{2}$, and the corresponding eigenvectors are the columns of the above matrix, namely

$$u_1 = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} \qquad u_2 = \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix} \qquad u_3 = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}.$$

We therefore have a diagonalisation $A = UDU^{-1}$, where

$$U = \begin{bmatrix} 1 & 0 & -1 \\ 2 & -1 & 1 \\ 1 & 1 & 0 \end{bmatrix} \qquad D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1/4 & 0 \\ 0 & 0 & 1/2 \end{bmatrix}.$$

We can calculate the inverse of $U$ by entering `U^(-1)` in Maple; we find that

$$U^{-1} = \frac{1}{4} \begin{bmatrix} 1 & 1 & 1 \\ -1 & -1 & 3 \\ -3 & 1 & 1 \end{bmatrix}$$

This gives

$$\lim_{n\to\infty} D^n = \lim_{n\to\infty} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1/4^n & 0 \\ 0 & 0 & 1/2^n \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

We will call this matrix $D^\infty$. As $A^n = UD^nU^{-1}$ we see that

$$\lim_{n\to\infty} A^n = UD^\infty U^{-1} = \frac{1}{4} \begin{bmatrix} 1 & 0 & -1 \\ 2 & -1 & 1 \\ 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ -1 & -1 & 3 \\ -3 & 1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 0.25 & 0 & -0.25 \\ 0.5 & -0.25 & 0.25 \\ 0.25 & 0.25 & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0.25 & 0.25 & 0.25 \\ 0.5 & 0.5 & 0.5 \\ 0.25 & 0.25 & 0.25 \end{bmatrix}.$$

As a check, we can enter `evalf(A^10)` in Maple to calculate a numerical approximation to $A^{10}$, which is

$$\begin{bmatrix} 0.2507324219 & 0.2497558594 & 0.2497558594 \\ 0.4992678165 & 0.5002443790 & 0.5002434254 \\ 0.2499997616 & 0.2499997616 & 0.2500007153 \end{bmatrix}.$$

This is already quite close to the limiting value.

**Exercise 9.** Consider the matrix

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}.$$

You may assume that this matrix cannot be diagonalised. Nonetheless, the powers $A^n$ follow a simple pattern. Calculate $A^n$ for some small values of $n$, then see if you can find the general rule, then prove it by induction.

**Solution:** The first few powers are as follows:

$$A^0 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad A^1 = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

$$A^2 = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix} \qquad A^3 = \begin{bmatrix} 1 & 3 & 6 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{bmatrix}$$

$$A^4 = \begin{bmatrix} 1 & 4 & 10 \\ 0 & 1 & 4 \\ 0 & 0 & 1 \end{bmatrix} \qquad A^5 = \begin{bmatrix} 1 & 5 & 15 \\ 0 & 1 & 5 \\ 0 & 0 & 1 \end{bmatrix}.$$

From this it is at least clear that

$$A^n = \begin{bmatrix} 1 & n & p_n \\ 0 & 1 & n \\ 0 & 0 & 1 \end{bmatrix}$$

for some number $p_n$. The remaining problem is to find a formula for $p_n$. The first few cases are

$$p_0 = 0 \qquad p_1 = 1 \qquad p_2 = 3 \qquad p_3 = 6 \qquad p_4 = 10 \qquad p_5 = 15.$$

You might recognise these numbers as coming from Pascal's triangle, or you might notice that $p_n - p_{n-1} = n$ and work from there, or you might notice that $p_n$ is approximately $n^2/2$ and so study $p_n - n^2/2$, or you might enter the above numbers in the Online Encyclopedia of Integer Sequences at `http://oeis.org` and see what it finds. By any of these means you can arrive at the formula

$$p_n = \binom{n+1}{2} = (n^2 + n)/2.$$

We thus conclude that

$$A^n = \begin{bmatrix} 1 & n & (n^2+n)/2 \\ 0 & 1 & n \\ 0 & 0 & 1 \end{bmatrix}.$$

We can prove this formally by induction. The claim is clearly true for $n = 0$. If it holds for a particular value of $n$, then we have

$$\begin{aligned} A^{n+1} = AA^n &= \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & n & (n^2+n)/2 \\ 0 & 1 & n \\ 0 & 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 1 & n+1 & (n^2+n)/2 + n + 1 \\ 0 & 1 & n+1 \\ 0 & 0 & 1 \end{bmatrix}. \end{aligned}$$

Here

$$(n^2 + n)/2 + n + 1 = n^2/2 + 3n/2 + 1 = ((n+1)^2 + (n+1))/2,$$

so we see that the claim also holds for $n + 1$. Thus, by induction, it holds for all natural numbers $n$.