MAS201 PROBLEM SHEET 6

Lecture 11

Exercise 1. Solve the following system of differential equations using the method in Section 15:

$$\dot{x} = -2x - 3y \qquad x = 1 \text{ when } t = 0$$
$$\dot{y} = 3x - 2y \qquad y = 0 \text{ when } t = 0.$$

This involves complex eigenvalues. You should remember the rules

$$\cos(t) = (e^{it} + e^{-it})/2 \qquad \sin(t) = (e^{it} - e^{-it})/(2i).$$

Solution: Put

$$u = \begin{bmatrix} x \\ y \end{bmatrix} \qquad A = \begin{bmatrix} -2 & -3 \\ 3 & -2 \end{bmatrix} \qquad c = \begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$

Our differential equation can be written in matrix form as $\dot{u} = Au$, with $u = c$ when $t = 0$. To solve it, we must find a diagonalisation $A = UDU^{-1}$, and then u will be $Ue^{Dt}U^{-1}c$.

The characteristic polynomial of A is

$$\chi_A(t) = \det \begin{bmatrix} -2-t & -3 \\ 3 & -2-t \end{bmatrix} = (t+2)^2 + 9.$$

For t to be a root we must have $(t+2)^2 = -9$, so $t + 2 = \pm 3i$. It follows that the eigenvalues are $\lambda_1 = 3i - 2$ and $\lambda_2 = -3i - 2$. To find an eigenvector $u_1 = \begin{bmatrix} a & b \end{bmatrix}^T$ of eigenvalue λ_1, we perform the row-reduction

$$A - \lambda_1 I = \begin{bmatrix} -3i & -3 \\ 3 & -3i \end{bmatrix} \to \begin{bmatrix} 1 & -i \\ 3 & -3i \end{bmatrix} \to \begin{bmatrix} 1 & -i \\ 0 & 0 \end{bmatrix}.$$

It follows that $a - ib = 0$, so $a = ib$. We can take $b = 1$ to get $u_1 = \begin{bmatrix} i & 1 \end{bmatrix}^T$.

Similarly, to find an eigenvector $u_2 = \begin{bmatrix} a & b \end{bmatrix}^T$ of eigenvalue λ_2, we perform the row-reduction

$$A - \lambda_2 I = \begin{bmatrix} 3i & -3 \\ 3 & 3i \end{bmatrix} \to \begin{bmatrix} 1 & i \\ 3 & 3i \end{bmatrix} \to \begin{bmatrix} 1 & i \\ 0 & 0 \end{bmatrix}.$$

It follows that $a + ib = 0$, so $a = -ib$. We can take $b = 1$ to get $u_2 = \begin{bmatrix} -i & 1 \end{bmatrix}^T$.

We now have a diagonalisation $A = UDU^{-1}$, where

$$U = \begin{bmatrix} i & -i \\ 1 & 1 \end{bmatrix} \qquad D = \begin{bmatrix} 3i-2 & 0 \\ 0 & -3i-2 \end{bmatrix}$$

This gives

$$\begin{aligned}
U^{-1} &= \frac{1}{2i} \begin{bmatrix} 1 & i \\ -1 & i \end{bmatrix} \\
e^{Dt} &= \begin{bmatrix} e^{(3i-2)t} & 0 \\ 0 & e^{(-3i-2)t} \end{bmatrix} \\
u = Ue^{Dt}U^{-1}c &= \frac{1}{2i} \begin{bmatrix} i & -i \\ 1 & 1 \end{bmatrix} \begin{bmatrix} e^{(3i-2)t} & 0 \\ 0 & e^{(-3i-2)t} \end{bmatrix} \begin{bmatrix} 1 & i \\ -1 & i \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} \\
&= \frac{1}{2i} \begin{bmatrix} ie^{(3i-2)t} & -ie^{(-3i-2)t} \\ e^{(3i-2)t} & e^{(-3i-2)t} \end{bmatrix} \begin{bmatrix} 1 \\ -1 \end{bmatrix} = e^{-2t} \begin{bmatrix} e^{3it}/2 + e^{-3it}/2 \\ e^{3it}/(2i) - e^{-3it}/(2i) \end{bmatrix} \\
&= e^{-2t} \begin{bmatrix} \cos(3t) \\ \sin(3t) \end{bmatrix}.
\end{aligned}$$

We thus have $x = e^{-2t}\cos(3t)$ and $y = e^{-2t}\sin(3t)$.

Exercise 2. Solve the following system of differential equations:

$$\begin{aligned}\dot{x}_1 &= 0.2x_1 + 0.5x_2 + 0.3x_3\\ \dot{x}_2 &= 0.6x_1 + 0.6x_2 + 0.7x_3\\ \dot{x}_3 &= 0.1x_1 + 0.4x_2 + 0.8x_3,\end{aligned}$$

with $x = \begin{bmatrix}1 & 0 & 0\end{bmatrix}^T$ at $t = 0$. You should use Maple to calculate the relevant eigenvalues and eigenvectors. Unlike most examples in this course, this one has not been fine-tuned to work out with nice round numbers.

Solution: We have $\dot{x} = Ax$ and $x = c$ at $t = 0$, where

$$A = \begin{bmatrix}0.2 & 0.5 & 0.3\\ 0.6 & 0.6 & 0.7\\ 0.1 & 0.4 & 0.8\end{bmatrix} \qquad c = \begin{bmatrix}1\\0\\0\end{bmatrix}.$$

The general method is to diagonalise A as UDU^{-1} with $D = \mathrm{diag}(\lambda_1, \lambda_2, \lambda_3)$ say, then $x = UEU^{-1}c$, where $E = \mathrm{diag}(e^{\lambda_1 t}, e^{\lambda_2 t}, e^{\lambda_3 t})$. We can do this in Maple as follows:

```
with(LinearAlgebra):
unprotect('D'):

A := <<0.2|0.5|0.3>,<0.6|0.6|0.7>,<0.1|0.4|0.8>>;
L,U := Eigenvectors(A);
D := DiagonalMatrix(L);
E := DiagonalMatrix([exp(L[1]*t),exp(L[2]*t),exp(L[3]*t)]);
c := <1,0,0>;
x := U . E . U^(-1);
```

Maple responds with

$$x := \begin{bmatrix}0.1471732926\,\mathrm{e}^{1.442698079\,t} + 0.5641142246\,\mathrm{e}^{-0.2096633632\,t} + 0.2887124828\,\mathrm{e}^{0.3669652806\,t}\\ 0.2563257383\,\mathrm{e}^{1.442698079\,t} - 0.5623411149\,\mathrm{e}^{-0.2096633632\,t} + 0.3060153766\,\mathrm{e}^{0.3669652806\,t}\\ 0.1824303322\,\mathrm{e}^{1.442698079\,t} + 0.1669120914\,\mathrm{e}^{-0.2096633632\,t} - 0.3493424236\,\mathrm{e}^{0.3669652806\,t}\end{bmatrix}$$

which is the solution for x. Some comments on these commands:

- Maple usually uses the symbol D for differentiataion, so if we want to use D as the name of a matrix, we need to enter `unprotect('D')` first. The quotation marks are important here.
- The line `L,U := Eigenvectors(A)` sets L to be the vector $\begin{bmatrix}\lambda_1 & \lambda_2 & \lambda_3\end{bmatrix}^T$, whose entries are the eigenvalues. It also sets U to be the usual matrix whose columns are the corresponding eigenvectors.

Exercise 3. Consider the matrix $A = \begin{bmatrix}1 & 1\\ 2 & 1\end{bmatrix}$.

(a) Find the eigenvalues of A.

(b) For each eigenvalue, find a corresponding eigenvector of A.

(c) Define recursively a sequence of vectors $\begin{bmatrix}p_n\\ q_n\end{bmatrix}$ as follows: we have $p_0 = 1$ and $q_0 = 0$, and for all $n > 0$ we have

$$\begin{aligned}p_n &= p_{n-1} + q_{n-1}\\ q_n &= 2p_{n-1} + q_{n-1}.\end{aligned}$$

Use your eigenvectors of A to find expressions for p_n and q_n (for a general positive integer n).

Note: this is not exactly like any of the examples in the notes. You will need to understand the theory behind those examples and adapt it slightly.

Solution:

(a) We have

$$\chi_A(t) = \det\begin{bmatrix}1-t & 1\\ 1 & 1-t\end{bmatrix} = (1-t)^2 - 2 = t^2 - 2t - 1 = \left(t - 1 - \sqrt{2}\right)\left(t - 1 + \sqrt{2}\right).$$

We thus see that the eigenvalues of A are $\lambda_1 = 1 + \sqrt{2}$ and $\lambda_2 = 1 - \sqrt{2}$.

(b) To find an eigenvector u_1 of eigenvalue λ_1, we perform the following row-reduction:

$$A - \lambda_1 I = \begin{bmatrix} -\sqrt{2} & 1 \\ 2 & -\sqrt{2} \end{bmatrix} \to \begin{bmatrix} 1 & -1/\sqrt{2} \\ 2 & -\sqrt{2} \end{bmatrix} \to \begin{bmatrix} 1 & -1/\sqrt{2} \\ 0 & 0 \end{bmatrix}.$$

We see that $u_1 = \begin{bmatrix} a & b \end{bmatrix}^T$ with $a - b/\sqrt{2} = 0$. It is convenient to take $b = \sqrt{2}$ giving $u_1 = \begin{bmatrix} 1 & \sqrt{2} \end{bmatrix}^T$.

Similarly, to find an eigenvector u_2 of eigenvalue λ_2, we perform the following row-reduction:

$$A - \lambda_2 I = \begin{bmatrix} \sqrt{2} & 1 \\ 2 & \sqrt{2} \end{bmatrix} \to \begin{bmatrix} 1 & 1/\sqrt{2} \\ 2 & \sqrt{2} \end{bmatrix} \to \begin{bmatrix} 1 & 1/\sqrt{2} \\ 0 & 0 \end{bmatrix}.$$

We see that $u_2 = \begin{bmatrix} a & b \end{bmatrix}^T$ with $a + b/\sqrt{2} = 0$. It is convenient to take $b = -\sqrt{2}$ giving $u_2 = \begin{bmatrix} 1 & -\sqrt{2} \end{bmatrix}^T$.

(c) Put $m_n = \begin{bmatrix} p_n \\ q_n \end{bmatrix}$. We are given that

$$m_0 = \begin{bmatrix} p_0 \\ q_0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$
$$m_n = \begin{bmatrix} p_n \\ q_n \end{bmatrix} = \begin{bmatrix} p_{n-1} + q_{n-1} \\ 2p_{n-1} + q_{n-1} \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} p_{n-1} \\ q_{n-1} \end{bmatrix} = Am_{n-1}.$$

Using the rule $m_n = Am_{n-1}$ repeatedly we see that $m_1 = Am_0$, so $m_2 = Am_1 = A^2 m_0$, so $m_3 = Am_2 = A^3 m_0$, and so on; so $m_n = A^n m_0$ for all n. To understand A^n, we note that the above eigenvectors give a diagonalisation $A = UDU^{-1}$, where

$$U = [u_1|u_2] = \begin{bmatrix} 1 & 1 \\ \sqrt{2} & -\sqrt{2} \end{bmatrix}$$
$$D = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} = \begin{bmatrix} 1+\sqrt{2} & 0 \\ 0 & 1-\sqrt{2} \end{bmatrix}$$
$$U^{-1} = \frac{1}{-2\sqrt{2}} \begin{bmatrix} -\sqrt{2} & -1 \\ -\sqrt{2} & 1 \end{bmatrix} = \begin{bmatrix} 1/2 & \sqrt{2}/4 \\ 1/2 & -\sqrt{2}/4 \end{bmatrix}.$$

This gives $A^n = UD^nU^{-1}$, so $m_n = UD^nU^{-1}m_0$, or

$$\begin{aligned} m_n &= \begin{bmatrix} 1 & 1 \\ \sqrt{2} & -\sqrt{2} \end{bmatrix} \begin{bmatrix} (1+\sqrt{2})^n & 0 \\ 0 & (1-\sqrt{2})^n \end{bmatrix} \begin{bmatrix} 1/2 & \sqrt{2}/4 \\ 1/2 & -\sqrt{2}/4 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} \\ &= \begin{bmatrix} (1+\sqrt{2})^n & (1-\sqrt{2})^n \\ (1+\sqrt{2})^n\sqrt{2} & -(1-\sqrt{2})^n\sqrt{2} \end{bmatrix} \begin{bmatrix} 1/2 \\ 1/2 \end{bmatrix} \\ &= \begin{bmatrix} ((1+\sqrt{2})^n + (1-\sqrt{2})^n)/2 \\ ((1+\sqrt{2})^n - (1-\sqrt{2})^n)/\sqrt{2} \end{bmatrix}. \end{aligned}$$

We thus have

$$p_n = ((1+\sqrt{2})^n + (1-\sqrt{2})^n)/2$$
$$q_n = ((1+\sqrt{2})^n - (1-\sqrt{2})^n)/\sqrt{2}.$$

Exercise 4. The sequence $(a_n)_{n=0}^{\infty}$ is given by $a_0 = 1001001$, $a_1 = 1010100$, $a_2 = 1110000$ and

$$a_{n+3} = 111a_{n+2} - 1110a_{n+1} + 1000a_n \qquad (\text{ for } n > 2)$$

(a) Write down a matrix equation relating the vector u_n to u_{n+1}, where $u_n = \begin{bmatrix} a_{n+2} \\ a_{n+1} \\ a_n \end{bmatrix}$.

(b) Find the eigenvalues and eigenvectors of the matrix occuring in (a). (If you have done this correctly, the answers will be integers with a nice pattern.)

(c) Express u_0 as a linear combination of the eigenvectors in (b).

(d) Give a general formula for a_n.
Hint: if you do this in the most obvious way, you will need to invert a certain matrix U to give a diagonalisation $A = UDU^{-1}$. However, part (c) provides a shortcut, so you do not need to find U^{-1}.

(e) Check directly that your formula satisfies $a_{n+3} = 111a_{n+2} - 1110a_{n+1} + 1000a_n$ and that a_0, a_1 and a_2 are as they should be.

Solution:

(a) We have

$$u_{n+1} = \begin{bmatrix} a_{n+3} \\ a_{n+2} \\ a_{n+1} \end{bmatrix} = \begin{bmatrix} 111a_{n+2} - 1110a_{n+1} + 1000a_n \\ a_{n+2} \\ a_{n+1} \end{bmatrix} = \begin{bmatrix} 111 & -1110 & 1000 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} a_{n+2} \\ a_{n+1} \\ a_n \end{bmatrix}.$$

In other words, if we put

$$A = \begin{bmatrix} 111 & -1110 & 1000 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

then $u_{n+1} = Au_n$. It follows that for all $n \geq 0$ we have

$$u_n = A^n u_0 = A^n \begin{bmatrix} 1110000 \\ 1010100 \\ 1001001 \end{bmatrix}.$$

(b) The characteristic polynomial is

$$\begin{aligned}
\chi_A(t) &= \det \begin{bmatrix} 111-t & -1110 & 1000 \\ 1 & -t & 0 \\ 0 & 1 & -t \end{bmatrix} = (111-t)\det\begin{bmatrix} -t & 0 \\ 1 & -t \end{bmatrix} + 1110\det\begin{bmatrix} 1 & 0 \\ 0 & -t \end{bmatrix} + 1000\det\begin{bmatrix} 1 & -t \\ 0 & 1 \end{bmatrix} \\
&= (111-t)t^2 - 1110t + 1000 = 1000 - 1110t + 111t^2 - t^3 \\
&= (1-t)(10-t)(100-t).
\end{aligned}$$

Thus, the eigenvalues are 1, 10 and 100. To find the corresponding eigenvectors, we perform the following row-reductions:

$$A - I = \begin{bmatrix} 110 & -1110 & 1000 \\ 1 & -1 & 0 \\ 0 & 1 & -1 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix} =: B_1$$

$$A - 10I = \begin{bmatrix} 101 & -1110 & 1000 \\ 1 & -10 & 0 \\ 0 & 1 & -10 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & -100 \\ 0 & 1 & -10 \\ 0 & 0 & 0 \end{bmatrix} =: B_2$$

$$A - 100I = \begin{bmatrix} 10 & -1110 & 1000 \\ 1 & -100 & 0 \\ 0 & 1 & -100 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & -10000 \\ 0 & 1 & -100 \\ 0 & 0 & 0 \end{bmatrix} =: B_3.$$

To find an eigenvector $w_2 = \begin{bmatrix} x & y & z \end{bmatrix}^T$ of eigenvalue 10, we need to solve $(A - 10I)w_2 = 0$, or equivalently $B_2 w_2 = 0$, which just reduces to $x = 100z$ and $y = 10z$ with z arbitrary. Taking $z = 1$, we see that $\begin{bmatrix} 100 & 10 & 1 \end{bmatrix}^T$ is an eigenvector of eigenvalue 10. Treating the other two eigenvalues in the same way, we find that the vectors

$$w_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \qquad w_2 = \begin{bmatrix} 100 \\ 10 \\ 1 \end{bmatrix} \qquad w_3 = \begin{bmatrix} 10000 \\ 100 \\ 1 \end{bmatrix}$$

are eigenvectors of eigenvalues 1, 10 and 100 respectively.

(c) By inspection we have

$$\begin{aligned}
u_0 &= \begin{bmatrix} 1110000 \\ 1010100 \\ 1001001 \end{bmatrix} = \begin{bmatrix} 1000000 \\ 1000000 \\ 1000000 \end{bmatrix} + \begin{bmatrix} 100000 \\ 10000 \\ 1000 \end{bmatrix} + \begin{bmatrix} 10000 \\ 100 \\ 1 \end{bmatrix} \\
&= 1000000 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + 1000 \begin{bmatrix} 100 \\ 10 \\ 1 \end{bmatrix} + \begin{bmatrix} 10000 \\ 100 \\ 1 \end{bmatrix} = 10^6 w_1 + 10^3 w_2 + w_3.
\end{aligned}$$

(d) Recall that $Aw_1 = w_1$ and $Aw_2 = 10w_2$ and $Aw_3 = 100w_3$. It follows that for all $n \geq 0$ we have $A^n w_1 = w_1$ and $A^n w_2 = 10^n w_2$ and $A^n w_3 = 100^n w_3 = 10^{2n} w_3$. This gives

$$\begin{aligned} u_n &= A^n u_0 = A^n(10^6 w_1 + 10^3 w_2 + w_3) = 10^6 A^n w_1 + 10^3 A^n w_2 + A^n w_3 \\ &= 10^6 w_1 + 10^{n+3} w_2 + 10^{2n} w_3 = 10^6 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + 10^{n+3} \begin{bmatrix} 10^2 \\ 10 \\ 1 \end{bmatrix} + 10^{2n} \begin{bmatrix} 10^4 \\ 10^2 \\ 1 \end{bmatrix} \\ &= \begin{bmatrix} 10^6 + 10^{n+5} + 10^{2n+4} \\ 10^6 + 10^{n+4} + 10^{2n+2} \\ 10^6 + 10^{n+3} + 10^{2n} \end{bmatrix}. \end{aligned}$$

In particular, a_n is the bottom entry in u_n, which is

$$a_n = 10^6 + 10^{n+3} + 10^{2n}.$$

(e) Our formula gives

$$\begin{aligned} a_0 &= 10^6 + 10^3 + 10^0 = 1001001 \\ a_1 &= 10^6 + 10^4 + 10^2 = 1010100 \\ a_2 &= 10^6 + 10^5 + 10^4 = 1110000 \end{aligned}$$

as it should. We also have

$$\begin{aligned} &111a_{n+2} - 1110a_{n+1} + 1000a_n \\ =&111(10^6 + 10^{n+5} + 10^{2n+4}) - 1110(10^6 + 10^{n+4} + 10^{2n+2}) + 1000(10^6 + 10^{n+3} + 10^{2n}) \\ =&10^6(111 - 1110 + 1000) + 10^{n+3}(11100 - 11100 + 1000) + 10^{2n}(1110000 - 111000 + 1000) \\ =&10^6 + 1000 \times 10^{n+3} + 1000000 \times 10^{2n} = 10^6 + 10^{n+6} + 10^{2n+6} = a_{n+3}. \end{aligned}$$

Exercise 5. Let (a_n) be the sequence given by $a_0 = 2$ and $a_1 = 4$ and $a_{n+2} = 4a_{n+1} - a_n$ for $n \geq 0$. Give a general formula for a_n.
Hint: it is easiest to take a slight shortcut as in step (c) of the previous example.

Solution: The vectors $v_n = \begin{bmatrix} a_n \\ a_{n+1} \end{bmatrix}$ satisfy $v_0 = \begin{bmatrix} 2 \\ 4 \end{bmatrix}$ and

$$v_{n+1} = \begin{bmatrix} a_{n+1} \\ a_{n+2} \end{bmatrix} = \begin{bmatrix} a_{n+1} \\ 4a_{n+1} - a_n \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 4 \end{bmatrix} \begin{bmatrix} a_n \\ a_{n+1} \end{bmatrix} = Av_n,$$

where $A = \begin{bmatrix} 0 & 1 \\ -1 & 4 \end{bmatrix}$. It follows that $v_k = A^k v_0$ for all $k \geq 0$. To understand this more explicitly, we need to find the eigenvalues and eigenvectors of A. The characteristic polynomial is

$$\chi_A(t) = \det \begin{bmatrix} -t & 1 \\ -1 & 4-t \end{bmatrix} = -t(4-t) - (-1) = t^2 - 4t + 1.$$

The eigenvalues of A are the roots of $\chi_A(t)$, which are $\lambda_1 = (4 + \sqrt{16-4})/2 = 2 + \sqrt{3}$ and $\lambda_2 = 2 - \sqrt{3}$. We next want to find an eigenvector $u_1 = \begin{bmatrix} x \\ y \end{bmatrix}$ with $Au_1 = \lambda_1 u_1$, or in other words

$$\begin{bmatrix} 0 & 1 \\ -1 & 4 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \lambda_1 x \\ \lambda_1 y \end{bmatrix}$$

or $y = \lambda_1 x$ and $4y - x = \lambda_1 y$. If we substitute $y = \lambda_1 x$ then the equation $4y - x = \lambda_1 y$ becomes $4\lambda_1 x - x = \lambda_1^2 x$ or $(\lambda_1^2 - 4\lambda_1 + 1)x = 0$, which holds automatically because λ_1 is a root of $t^2 - 4t + 1 = 0$. It follows that we can take $u_1 = \begin{bmatrix} 1 \\ \lambda_1 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 + \sqrt{3} \end{bmatrix}$. Similarly, the vector $u_2 = \begin{bmatrix} 1 \\ \lambda_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 - \sqrt{3} \end{bmatrix}$ is an eigenvector of A with eigenvalue λ_2.

The best way to proceed from here is to observe that $v_0 = \begin{bmatrix} 2 \\ 4 \end{bmatrix} = u_1 + u_2$. We now have $A^n u_i = \lambda_i^n u_i$, so

$$\begin{aligned} v_n &= A^n v_0 = A^n(u_1 + u_2) = A^n u_1 + A^n u_2 = \lambda_1^n u_1 + \lambda_2^n u_2 \\ &= \begin{bmatrix} \lambda_1^n + \lambda_2^n \\ \lambda_1^{n+1} + \lambda_2^{n+1} \end{bmatrix}. \end{aligned}$$

On the other hand, we have $v_n = \begin{bmatrix} a_n \\ a_{n+1} \end{bmatrix}$, so we conclude that $a_n = \lambda_1^n + \lambda_2^n = (2+\sqrt{3})^n + (2-\sqrt{3})^n$.

We now explain a less efficient method that follows the lecture notes more closely. We have a diagonalisation $A = UDU^{-1}$, where

$$U = [u_1|u_2] = \begin{bmatrix} 1 & 1 \\ 2+\sqrt{3} & 2-\sqrt{3} \end{bmatrix} \qquad D = \begin{bmatrix} 2+\sqrt{3} & 0 \\ 0 & 2-\sqrt{3} \end{bmatrix} \qquad U^{-1} = \frac{1}{-2\sqrt{3}} \begin{bmatrix} 2-\sqrt{3} & -1 \\ -2-\sqrt{3} & 1 \end{bmatrix}.$$

This gives $A^n = UD^nU^{-1}$, so $v_n = UD^nU^{-1}v_0$, or

$$\begin{aligned} v_n &= \frac{1}{-2\sqrt{3}} \begin{bmatrix} 1 & 1 \\ 2+\sqrt{3} & 2-\sqrt{3} \end{bmatrix} \begin{bmatrix} (2+\sqrt{3})^n & 0 \\ 0 & (2-\sqrt{3})^n \end{bmatrix} \begin{bmatrix} 2-\sqrt{3} & -1 \\ -2-\sqrt{3} & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 4 \end{bmatrix} \\ &= \frac{1}{-2\sqrt{3}} \begin{bmatrix} (2+\sqrt{3})^n & (2-\sqrt{3})^n \\ (2+\sqrt{3})^{n+1} & (2-\sqrt{3})^{n+1} \end{bmatrix} \begin{bmatrix} -2\sqrt{3} \\ -2\sqrt{3} \end{bmatrix} \\ &= \begin{bmatrix} (2+\sqrt{3})^n + (2-\sqrt{3})^n \\ (2+\sqrt{3})^{n+1} + (2-\sqrt{3})^{n+1} \end{bmatrix}, \end{aligned}$$

just as before.

Lecture 12

Exercise 6. Over a period of 5 minutes, in a typical MAS201 lecture, 90% of students who are awake at the beginning of the 5-minute period will still be so at the end of it (but the other 10% will fall asleep) and 90% of students who are asleep at the beginning of the 5-minute period will still be so at the end of it (and the other 10% will wake up). If all the students are awake at the beginning of the lecture, what percentage will be awake at the end of the lecture, 50 minutes later?

Solution: This process can be modelled as a Markov chain, with students in state 1 being awake, and students in state 2 being asleep. We are told that after a single period, students who start off awake (in state 1) will still be in state 1 with probability 90% = 0.9. In other words, we have $p_{1\leftarrow 1} = 0.9$. Similarly, we are told that $p_{2\leftarrow 2} = 0.9$ and $p_{1\leftarrow 2} = p_{2\leftarrow 1} = 0.1$. Thus, the transition matrix is

$$P = \begin{bmatrix} 0.9 & 0.1 \\ 0.1 & 0.9 \end{bmatrix}.$$

As all students are awake at the beginning, the initial distribution is $r_0 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$. We need to find the distribution after 50 minutes (ie 10 periods of 5 minutes) which is $r_{10} = P^{10}r_0$. To evaluate this, we must diagonalise P. The characteristic polynomial is

$$\chi_P(t) = \det \begin{bmatrix} 0.9-t & 0.1 \\ 0.1 & 0.9-t \end{bmatrix} = t^2 - 1.8t + 0.8 = (t-1)(t-0.8).$$

Thus, the eigenvalues are $\lambda_1 = 1$ and $\lambda_2 = 0.8$. (Recall here that the transition matrix of a Markov chain is always stochastic, and any stochastic matrix has 1 as an eigenvalue.) It is easy to see that the corresponding eigenvectors are

$$u_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \qquad u_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}.$$

This gives a diagonalisation $P = UDU^{-1}$, where

$$U = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \qquad D = \begin{bmatrix} 1 & 0 \\ 0 & 0.8 \end{bmatrix} \qquad U^{-1} = \frac{1}{-2} \begin{bmatrix} -1 & -1 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} 0.5 & 0.5 \\ 0.5 & -0.5 \end{bmatrix}.$$

We therefore have $r_n = P^n r_0 = UD^nU^{-1}r_0$, ro

$$\begin{aligned} r_n &= \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & (0.8)^n \end{bmatrix} \begin{bmatrix} 0.5 & 0.5 \\ 0.5 & -0.5 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} \\ &= \begin{bmatrix} 1 & (0.8)^n \\ 1 & -(0.8)^n \end{bmatrix} \begin{bmatrix} 0.5 \\ 0.5 \end{bmatrix} = \begin{bmatrix} 0.5(1+(0.8)^n) \\ 0.5(1+(0.8)^n) \end{bmatrix}. \end{aligned}$$

The proportion of students who are awake (in state 1) at the end is the first component of r_{10}, which is $0.5(1+(0.8)^10) \simeq 0.5537$. Thus, approximately 55.37% of students will be awake.

Exercise 7. Put $d = \begin{bmatrix} 1 & \cdots & 1 \end{bmatrix}^T \in \mathbb{R}^n$.

(a) If $P \in M_n(\mathbb{R})$ is a stochastic matrix, show that $d^TP = d^T$.

(b) Deduce that if $q \in \mathbb{R}^n$ is a probability vector, then Pq is also a probability vector.
(c) Deduce that if $Q \in M_n(\mathbb{R})$ is another stochastic matrix, then PQ is also a stochastic matrix. (**Hint:** how are the columns of PQ related to the columns of Q?)

Solution:

(a) Let the columns of P be $v_1, \dots, v_n$. As P is stochastic, we know that the sum of the entries in v_i is equal to one, so $d.v_i = 1$. This means that

$$d^T P = \begin{bmatrix} 1 & \cdots & 1 \end{bmatrix} \left[\begin{array}{c|c|c} v_1 & \cdots & v_n \end{array} \right] = \begin{bmatrix} d.v_1 & \cdots & d.v_n \end{bmatrix} = \begin{bmatrix} 1 & \cdots & 1 \end{bmatrix} = d^T.$$

(b) Now let q be a probability vector. Then all entries in P and q are nonnegative, and the entries in Pq are sums of entries in P multiplied by entries in q, so they are again nonnegative. Moreover, the sum of the entries in Pq is $d.Pq = d^T Pq$, but $d^T P = d$, so this is the same as $d^T q = d.q$, which is one by assumption. This proves that Pq is a probability vector.

(c) Now let Q be another $n \times n$ stochastic matrix. Let $w_1, \dots, w_n$ be the columns of Q, which are probability vectors. We then have

$$PQ = P \left[\begin{array}{c|c|c} w_1 & \cdots & w_n \end{array} \right] = \left[\begin{array}{c|c|c} Pw_1 & \cdots & Pw_n \end{array} \right].$$

The vectors $Pw_1, \dots, Pw_n$ are probability vectors by part (b), and it follows that PQ is a stochastic matrix.

Exercise 8. Suppose that $0 < p, q < 1$, and put $P = \begin{bmatrix} p & 1-q \\ 1-p & q \end{bmatrix}$ (so P is a stochastic matrix). Find the eigenvalues and eigenvectors of P in terms of p and q.
(**Hint:** a general theorem from lectures tells you one of the eigenvalues.)

Solution: The characteristic polynomial is

$$\chi_P(t) = \det \begin{bmatrix} p-t & 1-q \\ 1-p & q-t \end{bmatrix} = (p-t)(q-t) - (1-p)(1-q) = t^2 - (p+q)t + (p+q-1).$$

Every stochastic matrix has 1 as an eigenvalue, so one of the roots of $\chi_P(t)$ is at $t = 1$. We can divide $t^2 - (p+q)t - (1-p-q)$ by $t-1$ to obtain the factorisation $\chi_P(t) = (t-1)(t-(p+q-1))$, so the other eigenvalue is $r = p+q-1$. To find an eigenvector $u_1 = \begin{bmatrix} x \\ y \end{bmatrix}$ of eigenvalue 1, we must solve

$$(P-I)u_1 = \begin{bmatrix} p-1 & 1-q \\ 1-p & q-1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

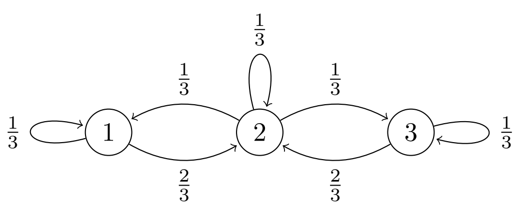
This reduces to $(1-p)x = (1-q)y$ so we can take $y = 1/(1-q)$ to get $x = 1/(1-p)$ and $u_1 = \begin{bmatrix} 1/(1-p) \\ 1/(1-q) \end{bmatrix}$.

Next, to find an eigenvector of eigenvalue r we note that

$$P - rI = \begin{bmatrix} p & 1-q \\ 1-p & q \end{bmatrix} - \begin{bmatrix} p+q-1 & 0 \\ 0 & p+q-1 \end{bmatrix} = \begin{bmatrix} 1-q & 1-q \\ 1-p & 1-p \end{bmatrix}.$$

It follows that the vector $u_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ satisfies $(P-rI)u_2 = 0$, so this is the required eigenvector.

Exercise 9. Consider the following Markov chain:



Write down the transition matrix and find its eigenvalues and eigenvectors. What is the stationary distribution?

Solution: The transition matrix is

$$P = \begin{bmatrix} p_{1\leftarrow1} & p_{1\leftarrow2} & p_{1\leftarrow3} \\ p_{2\leftarrow1} & p_{2\leftarrow2} & p_{2\leftarrow3} \\ p_{3\leftarrow1} & p_{3\leftarrow2} & p_{3\leftarrow3} \end{bmatrix} = \begin{bmatrix} 1/3 & 1/3 & 0 \\ 2/3 & 1/3 & 2/3 \\ 0 & 1/3 & 1/3 \end{bmatrix}.$$

For the characteristic polynomial, we have

$$\begin{aligned}
\chi_P(t) &= \det \begin{bmatrix} 1/3-t & 1/3 & 0 \\ 2/3 & 1/3-t & 2/3 \\ 0 & 1/3 & 1/3-t \end{bmatrix} \\
&= (1/3-t)\det \begin{bmatrix} 1/3-t & 2/3 \\ 1/3 & 1/3-t \end{bmatrix} - (1/3)\det \begin{bmatrix} 2/3 & 2/3 \\ 0 & 1/3-t \end{bmatrix} \\
\det \begin{bmatrix} 1/3-t & 2/3 \\ 1/3 & 1/3-t \end{bmatrix} &= (1/3-t)^2 - 2/9 = t^2 - (2/3)t - 1/9 \\
\det \begin{bmatrix} 2/3 & 2/3 \\ 0 & 1/3-t \end{bmatrix} &= 2/9 - (2/3)t \\
\chi_P(t) &= (1/3-t)(t^2-(2/3)t-1/9) - (1/3)(2/9-(2/3)t) \\
&= -1/9 + (1/9)t + t^2 - t^3 = (1-t)(t^2-1/9) = (1-t)(t-1/3)(t+1/3).
\end{aligned}$$

From this we see that the eigenvalues are $1/3$, $-1/3$ and 1. To find an eigenvector u_1 of eigenvalue $1/3$ we row-reduce $P - \frac{1}{3}I$:

$$\begin{bmatrix} 0 & 1/3 & 0 \\ 2/3 & 0 & 2/3 \\ 0 & 1/3 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

This means that if $u_1 = \begin{bmatrix} x & y & z \end{bmatrix}^T$ we must have

$$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

which gives $x = -z$ with $y = 0$. Taking $z = 1$ we get $u_1 = \begin{bmatrix} -1 & 0 & 1 \end{bmatrix}^T$. Next, to find an eigenvector u_2 of eigenvalue $-1/3$ we row-reduce $P + \frac{1}{3}I$:

$$\begin{bmatrix} 2/3 & 1/3 & 0 \\ 2/3 & 2/3 & 2/3 \\ 0 & 1/3 & 2/3 \end{bmatrix} \to \begin{bmatrix} 1 & 1/2 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 2 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}.$$

This means that if $u_2 = \begin{bmatrix} x & y & z \end{bmatrix}^T$ we must have

$$\begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

which gives $x = z$ and $y = -2z$. Taking $z = 1$ we get $u_2 = \begin{bmatrix} 1 & -2 & 1 \end{bmatrix}^T$. Finally, to find an eigenvector of eigenvalue 1 we row-reduce $P - I$:

$$\begin{bmatrix} -2/3 & 1/3 & 0 \\ 2/3 & -2/3 & 2/3 \\ 0 & 1/3 & -2/3 \end{bmatrix} \to \begin{bmatrix} 1 & -1/2 & 0 \\ 1 & -1 & 1 \\ 0 & 1 & -2 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \end{bmatrix}.$$

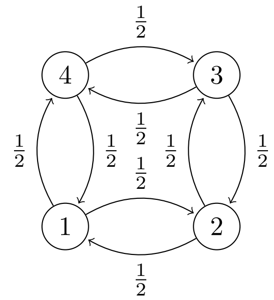
This means that if $u_3 = \begin{bmatrix} x & y & z \end{bmatrix}^T$ we must have

$$\begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

which gives $x = z$ and $y = 2z$. Taking $z = 1$, we get $u_3 = \begin{bmatrix} 1 & 2 & 1 \end{bmatrix}^T$.

We are also asked for a stationary distribution, which should be an eigenvector of eigenvalue 1 that is also a probability vector. To make u_3 into a probability vctor we need to divide it by 4, giving $\begin{bmatrix} \frac{1}{4} & \frac{1}{2} & \frac{1}{4} \end{bmatrix}^T$ as the stationary distribution.

Exercise 10. Consider the following Markov chain:



Write down the transition matrix P and check that $P^3 = P$. Deduce that $P^{2k+1} = P$ for all $k \geq 0$. If we start in state 1 at $t = 0$, what is the probability of being in state 3 at $t = 1111$?

Note: you do not need to calculate any eigenvalues or eigenvectors for this question.

Solution: The transition matrix is

$$P = \begin{bmatrix} p_{1\leftarrow 1} & p_{1\leftarrow 2} & p_{1\leftarrow 3} & p_{1\leftarrow 4} \\ p_{2\leftarrow 1} & p_{2\leftarrow 2} & p_{2\leftarrow 3} & p_{2\leftarrow 4} \\ p_{3\leftarrow 1} & p_{3\leftarrow 2} & p_{3\leftarrow 3} & p_{3\leftarrow 4} \\ p_{4\leftarrow 1} & p_{4\leftarrow 2} & p_{4\leftarrow 3} & p_{4\leftarrow 4} \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix}.$$

This gives

$$P^2 = \frac{1}{4} \begin{bmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix}$$

$$P^3 = \frac{1}{4} \begin{bmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix} = P.$$

We can now multiply both sides of the equation $P^3 = P$ by P^2 to get $P^5 = P^3$, but $P^3 = P$ so $P^5 = P$. We now multiply both sides by P^2 again to get $P^7 = P^3 = P$, and again to get $P^9 = P^3 = P$ and so on. Ths shows that $P^{2k+1} = P$ for all $k \geq 0$.

Now suppose we are definitely in state 1 at $t = 0$, so the distribution vector r_0 is $\begin{bmatrix} 1 & 0 & 0 & 0 \end{bmatrix}^T$. The distribution at $t = 1111$ is then $r_{1111} = P^{1111} r_0$, but we have just seen that $P^{1111} = P$, so

$$r_{1111} = P r_0 = \frac{1}{2} \begin{bmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 1/2 \\ 0 \\ 1/2 \end{bmatrix}.$$

By looking at the third entry, we see that the probability of being in state 3 at $t = 1111$ is zero. In fact, this can be seen even more directly. From the diagram we see that every second we switch from an odd-numbered state to an even numbered state or *vice-versa*. We start in state 1 at $t = 0$, and at $t = 1111$ we have switched over an odd number of times, so we must be in an even-numbered state, and in particular we cannot be in state 3.