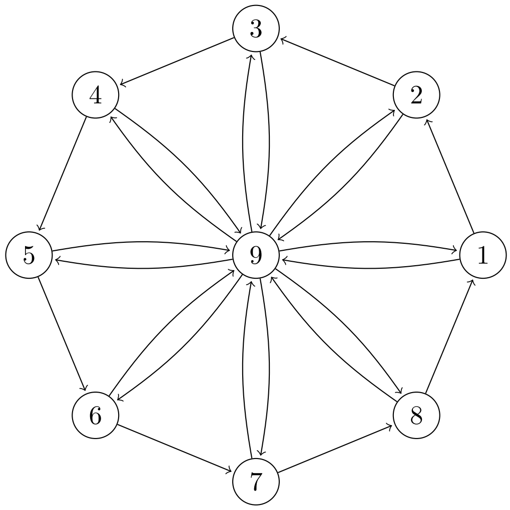
MAS201 PROBLEM SHEET 7

Lecture 13

Exercise 1. Consider the following web of pages and links.



Let a be the PageRank of page 1, and let b be the PageRank of page 9. By symmetry, pages 2 to 8 must also have rank a. Use the consistency and normalisation conditions to find a and b (without writing down any 9×9 matrices).

Solution: First, the normalisation condition says that $\sum_{i=1}^{9} r_i = 1$. As $r_1 = \cdots = r_8 = a$ and $r_9 = b$, this means that $8a + b = 1$.

Next, note that the numbers of outgoing links are $N_1 = \cdots = N_8 = 2$ and $N_9 = 8$. As page 1 has links from pages 8 and 9, the consistency condition says that $r_1 = r_8/N_8 + r_9/N_9$, or in other words $a = a/2 + b/8$. By symmetry, pages 2 to 8 have the same consistency condition as page 1. On the other hand, page 9 has links from pages 1 to 8, so the consistency condition there is

$$b = r_9 = r_1/N_1 + \cdots + r_8/N_8 = a/2 + \cdots + a/2 = 4a.$$

Solving the equations $8a + b = 1$, $a = a/2 + b/8$ and $b = 4a$ gives $a = 1/12$ and $b = 1/3$.

Lecture 14

Exercise 2. Each of the following sets is a subspace of $\mathbb{R}^n$ for suitable n.

(a) U_0 is the set of vectors $u = \begin{bmatrix} x & y & z \end{bmatrix}^T$ in $\mathbb{R}^3$ that satisfy $2x - y + 3z = 0$.

(b) U_1 is the set of vectors in $\mathbb{R}^4$ of the form $\begin{bmatrix} s & t-3s & t+2s & t-s \end{bmatrix}^T$.

(c) U_2 is the set of vectors in $\mathbb{R}^4$ that can be expressed as a linear combination of the vectors $a = \begin{bmatrix} 1 & 1 & 1 & 1 \end{bmatrix}^T$ and $b = \begin{bmatrix} 1 & 0 & 0 & 1 \end{bmatrix}^T$.

(d) U_3 is the set of vectors in $\mathbb{R}^3$ that are perpendicular to the vector $c = \begin{bmatrix} 5 & 6 & 7 \end{bmatrix}^T$.

Find two vectors p and q that both lie in U_0. Of course, there are many different answers for this that are equally correct. You should choose your vectors p and q such that they are nonzero and different from each other. Check that $p + q$ is an element of U_0. Then choose examples in the same way for U_1, U_2 and U_3.

Solution:

(a) One possibility is to take $p = \begin{bmatrix} 1 & 2 & 0 \end{bmatrix}^T$ (which lies in U_0 because $2 \times 1 - 2 + 3 \times 0 = 0$) and $q = \begin{bmatrix} 0 & 3 & 1 \end{bmatrix}^T$. We then have $p+q = \begin{bmatrix} 1 & 5 & 1 \end{bmatrix}^T$, which lies in U_0 because $2 \times 1 - 5 + 2 \times 1 = 0$.

(b) If we take $s = 1$ and $t = 0$ in the defining formula for U_1, we get the vector $\begin{bmatrix}1 & -3 & 2 & -1\end{bmatrix}^T$. We call this p, so $p \in U_1$. If we instead take $s = 0$ and $t = 1$ in the defining formula for U_1, we get the vector $\begin{bmatrix}0 & 1 & 1 & 1\end{bmatrix}^T$. We call this q, so $q \in U_1$. We then have $p+q = \begin{bmatrix}1 & -2 & 3 & 0\end{bmatrix}^T$. This can be written as $p+q = \begin{bmatrix}s & t-3s & t+2s & t-s\end{bmatrix}^T$ with $s = t = 1$, so $p+q \in U_1$.

(c) Here the obvious thing to do is to take

$$p = a = 1a + 0b = \begin{bmatrix}1 & 1 & 1 & 1\end{bmatrix}^T$$
$$q = b = 0a + 1b = \begin{bmatrix}1 & 0 & 0 & 1\end{bmatrix}^T.$$

Then p and q are elements of U_2. We have $p+q = \begin{bmatrix}2 & 1 & 1 & 2\end{bmatrix}^T = a+b$, and this is a linear combination of a and b, so $p+q \in U_2$ as expected.

(d) By the standard dot product test, a vector $u = \begin{bmatrix}x & y & z\end{bmatrix}^T$ is perpendicular to the vector $c = \begin{bmatrix}5 & 6 & 7\end{bmatrix}^T$ if and only if $u.c = 0$, or in other words $5x + 6y + 7z = 0$. We can thus take $p = \begin{bmatrix}6 & -5 & 0\end{bmatrix}^T$ and $q = \begin{bmatrix}0 & 7 & -6\end{bmatrix}$; these satisfy $p.c = q.c = 0$, so they are both elements of U_3. We then have $p+q = \begin{bmatrix}6 & 2 & -6\end{bmatrix}$. As $(p+q).c = 6 \times 5 + 2 \times 6 - 6 \times 7 = 0$, we see that $p+q$ is again an element of U_3.

Exercise 3. Consider the following sets

$$P_0 = \{\begin{bmatrix}x & y\end{bmatrix}^T \in \mathbb{R}^2 \mid x^2 \geq 1\}$$
$$P_1 = \{\begin{bmatrix}x & y\end{bmatrix}^T \in \mathbb{R}^2 \mid xy \geq 0\}$$
$$P_2 = \{\begin{bmatrix}x & y\end{bmatrix}^T \in \mathbb{R}^2 \mid y \leq x^2\}$$
$$P_3 = \{\begin{bmatrix}x & y\end{bmatrix}^T \in \mathbb{R}^2 \mid x+y \text{ is an integer }\}$$
$$P_4 = \{\begin{bmatrix}x & y\end{bmatrix}^T \in \mathbb{R}^2 \mid x^2 + y^2 \leq 1\}$$

The set P_0 is not closed under addition, because the vectors $u_0 = \begin{bmatrix}1\\0\end{bmatrix}$ and $u_1 = \begin{bmatrix}-1\\0\end{bmatrix}$ both lie in P_0, but the sum $u_0 + u_1 = \begin{bmatrix}0\\0\end{bmatrix}$ does not lie in P_0. Moreover, the set P_0 is not closed under scalar multiplication, because the vector $u_2 = \begin{bmatrix}1\\1\end{bmatrix}$ lies in P_0, but the product $0.5u_2 = \begin{bmatrix}0.5\\0.5\end{bmatrix}$ does not lie in P_0. Give similarly specific examples to show that

(a) P_1 is not closed under addition.
(b) P_2 is not closed under addition.
(c) P_2 is not closed under scalar multiplication.
(d) P_3 is not closed under scalar multiplication.
(e) P_4 is not closed under scalar multiplication.

Solution:

(a) P_1 contains the vectors $u_3 = \begin{bmatrix}0\\1\end{bmatrix}$ and $u_4 = \begin{bmatrix}-1\\0\end{bmatrix}$ but not the sum $u_3 + u_4 = \begin{bmatrix}-1\\1\end{bmatrix}$.

(b) P_2 contains the vectors $u_5 = \begin{bmatrix}1\\1\end{bmatrix}$ and $u_6 = \begin{bmatrix}-1\\1\end{bmatrix}$ but not the sum $u_5 + u_6 = \begin{bmatrix}0\\2\end{bmatrix}$.

(c) P_2 contains the vector $u_7 = \begin{bmatrix}0\\-1\end{bmatrix}$ but not the vector $(-1)u_7 = \begin{bmatrix}0\\1\end{bmatrix}$.

(d) P_3 contains the vector $u_8 = \begin{bmatrix}1\\0\end{bmatrix}$ but not the vector $0.5u_8 = \begin{bmatrix}0.5\\0\end{bmatrix}$.

(e) P_4 contains the vector u_8 as above, but not the vector $2u_8 = \begin{bmatrix}2\\0\end{bmatrix}$.

Exercise 4. Which of the following sets is a subspace of $\mathbb{R}^4$?

(a) V_1 is the set of vectors of the form $\begin{bmatrix}s & t+s & t-s & t\end{bmatrix}^T$ (for some $s, t \in \mathbb{R}$).
(b) V_2 is the set of vectors of the form $\begin{bmatrix}t & t^2 & t^3 & t^4\end{bmatrix}^T$ (for some $t \in \mathbb{R}$).
(c) V_3 is the set of vectors $v = \begin{bmatrix}w & x & y & z\end{bmatrix}^T$ that satisfy $w + 10x + 100y + 1000z = 1$.

(d) V_4 is the set of vectors $v = \begin{bmatrix} w & x & y & z \end{bmatrix}^T$ that satisfy $w - x + y - z = 0$.

(e) V_5 is the set of vectors $v = \begin{bmatrix} w & x & y & z \end{bmatrix}^T$ that satisfy $(w - x)^2 + (y - z)^2 = 0$.

Explain your answers carefully.

Solution:

(a) The set V_1 is a subspace of $\mathbb{R}^4$. Indeed, if $v, v' \in V_1$ then we have $v = \begin{bmatrix} s & t+s & t-s & t \end{bmatrix}^T$ and $v' = \begin{bmatrix} s' & t'+s' & t'-s' & t' \end{bmatrix}^T$ for some $s, t, s', t' in \mathbb{R}$. This means that $v+v' = \begin{bmatrix} s'' & t''+s'' & t''-s'' & t'' \end{bmatrix}^T$, where $s'' = s+s'$ and $t'' = t+t'$. It follows that $v+v' \in V_1$, so V_1 is closed under addition. Similarly, if a is any scalar, we have $av = \begin{bmatrix} s^* & t^*+s^* & t^*-s^* & t^* \end{bmatrix}^T$, where $s^* = as$ and $t^* = at$. This shows that $av \in V_1$, so V_1 is closed under scalar multiplication. Finally, by taking $s = t = 0$ we see that the zero vector lies in V_1.

(b) The set V_2 is not a subspace of $\mathbb{R}^4$. Indeed, by taking $t = 1$ we see that the vector $v = \begin{bmatrix} 1 & 1 & 1 & 1 \end{bmatrix}^T$ lies in V_2, but the vector $2v = \begin{bmatrix} 2 & 2 & 2 & 2 \end{bmatrix}^T$ does not lie in V_2, so V_2 is not closed under scalar multiplication.

(c) The set V_3 is not a subspace of $\mathbb{R}^4$, because the zero vector does not satisfy $w + 10x + 100y + 1000z = 1$ and so is not an element of V_3.

(d) The set V_4 is a subspace of $\mathbb{R}^4$. Indeed, the zero vector $\begin{bmatrix} w & x & y & z \end{bmatrix}^T = \begin{bmatrix} 0 & 0 & 0 & 0 \end{bmatrix}^T$ satisfies $w - x + y - z$ and so $0 \in V_4$. If we have elements $v = \begin{bmatrix} w & x & y & z \end{bmatrix}^T$ and $v' = \begin{bmatrix} w' & x' & y' & z' \end{bmatrix}^T$ in V_4 then the we have $w - x + y - z = 0$ and $w' - x' + y' - z' = 0$. By adding these equations we see that $(w+w') - (x+x') + (y+y') - (z+z') = 0$, which shows that the sum $v + v'$ is again an element of V_4, so V_4 is closed under addition. A similar argument shows that it is closed under scalar multiplication.

(e) The set V_5 is also a subspace of $\mathbb{R}^4$, although this fact is slightly disguised by the way that we have defined it. Because all squares are nonnegative, we see that the only way $(w-x)^2 + (y-z)^2$ can be zero is if $w = x$ and $y = z$. This means that V_5 is the set of vectors of the form $\begin{bmatrix} s & s & t & t \end{bmatrix}^T$, which is a subspace by the same method that we used in part (a).

Exercise 5. (a) Give an example of a subset $U_0 \subseteq \mathbb{R}^2$ that contains zero and is closed under addition but is not closed under scalar multiplication.

(b) Give an example of a subset $U_1 \subseteq \mathbb{R}^2$ that contains zero and is closed under scalar multiplication but is not closed under addition.

(c) Suppose that U_2 is a nonempty subset of $\mathbb{R}^2$ that is closed under addition and scalar multiplication. Show that U_2 contains the zero vector.

(d) Let U_3 be a subspace of $\mathbb{R}^1 = \mathbb{R}$. Show that U_3 is either $\{0\}$ or all of $\mathbb{R}$.

Solution:

(a) The simplest example is

$$U_0 = \left\{ \begin{bmatrix} x \\ y \end{bmatrix} \in \mathbb{R}^2 \mid x, y \geq 0 \right\}.$$

This is not closed under scalar multiplication, because $\begin{bmatrix} 1 \\ 1 \end{bmatrix} \in U_0$ but $(-1) \begin{bmatrix} 1 \\ 1 \end{bmatrix} \notin U_0$.

(b) The simplest example is

$$U_1 = \left\{ \begin{bmatrix} x \\ y \end{bmatrix} \in \mathbb{R}^2 \mid xy = 0 \right\}.$$

This is not closed under addition, because $\begin{bmatrix} 1 \\ 0 \end{bmatrix} \in U_1$ and $\begin{bmatrix} 0 \\ 1 \end{bmatrix} \in U_1$ but $\begin{bmatrix} 1 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \notin U_1$.

(c) As U_2 is nonempty, we can choose a vector $u \in U_2$. As U_2 is closed under scalar multiplication, we can multiply the vector $u \in U_2$ by the scalar $0 \in \mathbb{R}$, and the result $0u$ will again be an element of U_2. Of course $0u$ is just the zero vector, so the zero vector is an element of U_2.

(d) Let U_3 be a subspace of $\mathbb{R}$. As it is a subspace, it must contain zero. If it does not contain anything else, then $U_3 = \{0\}$. Suppose instead that it does contain something else, so there is a nonzero element $u \in U_3$. Consider another element $v \in \mathbb{R}$. As we are working with elements of $\mathbb{R}^1$ which are just numbers, we can make sense of multiplication and division (which are not defined for vectors in $\mathbb{R}^2$ and beyond). We can thus express v as the product of the scalar v/u with the vector $u \in U_3$. (There is no problem with dividing by u, because we have assumed that

$u \neq 0$.) As U_3 is closed under scalar multiplication, the product $(v/u)u$ lies in U_3, or in other words $v \in U_3$. This works for all vectors $v \in \mathbb{R}^1$, so we have $U_3 = \mathbb{R}^1$.