MAS201 PROBLEM SHEET 8

Lecture 15

Exercise 1. Let V be the set of vectors of the form

$$v = \begin{bmatrix} 2p - q & q + r & 3p & r \end{bmatrix}^T$$

(where p, q, and r are arbitrary real numbers). Find a list of vectors whose span is V.

Solution: This is similar to examples 19.16 and 19.17. The general form for elements of V is

$$v = \begin{bmatrix} 2p-q \\ q+r \\ 3p \\ r \end{bmatrix} = p \begin{bmatrix} 2 \\ 0 \\ 3 \\ 0 \end{bmatrix} + q \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \end{bmatrix} + r \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix}.$$

In other words, the elements of V are all the possible linear combinations of the three vectors occuring in the above formula. This means that

$$V = \operatorname{span}\left(\begin{bmatrix} 2 \\ 0 \\ 3 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix} \right).$$

Exercise 2. Put

$$A = \begin{bmatrix} 1 & 6 & 8 \\ 7 & 2 & 3 \end{bmatrix}$$

and $V = \{v \in \mathbb{R}^3 \mid Av = 0\}$. Find a list of vectors whose annihilator is V.

Solution: This is an instance of Proposition 19.14: the space V is by definition the kernel of A, and that proposition tells us that the kernel is the annihilator of the transposed rows. Thus, if we put $a_1 = \begin{bmatrix} 1 & 6 & 8 \end{bmatrix}^T$ and $a_2 = \begin{bmatrix} 7 & 2 & 3 \end{bmatrix}^T$ then $V = \operatorname{ann}(a_1, a_2)$. This can also be seen quite easily without reference to Proposition 19.14. If $v = \begin{bmatrix} x & y & z \end{bmatrix}^T$ then

$$Av = \begin{bmatrix} 1 & 6 & 8 \\ 7 & 2 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} x + 6y + 8z \\ 7x + 2y + 3z \end{bmatrix} = \begin{bmatrix} a_1.v \\ a_2.v \end{bmatrix},$$

so v lies in V iff $Av = 0$ iff $a_1.v = a_2.v = 0$ iff v lies in $\operatorname{ann}(a_1, a_2)$; this means that $V = \operatorname{ann}(a_1, a_2)$ as before.

Exercise 3. Put

$$a_1 = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix} \qquad a_2 = \begin{bmatrix} 4 \\ 3 \\ 2 \\ 1 \end{bmatrix} \qquad u = \begin{bmatrix} 1 \\ -1 \\ -1 \\ 1 \end{bmatrix} \qquad v = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}.$$

(a) Does u lie in $\operatorname{ann}(a_1, a_2)$?
(b) Does v lie in $\operatorname{ann}(a_1, a_2)$?
(c) Does u lie in $\operatorname{span}(a_1, a_2)$?
(d) Does v lie in $\operatorname{span}(a_1, a_2)$?

Solution:

(a) Yes, we have $u.a_1 = 1 - 2 - 3 + 4 = 0$ and $u.a_2 = 4 - 3 - 2 + 1 = 0$, so $u \in \operatorname{ann}(a_1, a_2)$.
(b) No, we have $v.a_1 = 1 + 2 + 3 + 4 = 10 \neq 0$, so $v \notin \operatorname{ann}(a_1, a_2)$. (We also have $v.a_2 \neq 0$, but the fact that $v.a_1 \neq 0$ is already enough to show that $v \notin \operatorname{ann}(a_1, a_2)$, so we do not really need to consider $v.a_2$.)

(c) No, u cannot be written as a linear combination of a_1 and a_2, so it does not lie in $\text{span}(a_1, a_2)$. One way to check this is to use Method 7.6, which involves row-reducing the matrix $[a_1|a_2|u]$:

$$\begin{bmatrix} 1 & 4 & 1 \\ 2 & 3 & -1 \\ 3 & 2 & -1 \\ 4 & 1 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 4 & 1 \\ 0 & -5 & -3 \\ 0 & -10 & -4 \\ 0 & -15 & -5 \end{bmatrix} \to \begin{bmatrix} 1 & 4 & 1 \\ 0 & 1 & 0.6 \\ 0 & -10 & -4 \\ 0 & -15 & -5 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & -1.4 \\ 0 & 1 & 0.6 \\ 0 & 0 & 2 \\ 0 & 0 & 4 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}.$$

We end up with a pivot in the last column, which indicates that the equation $\lambda_1 a_1 + \lambda_2 a_2 = u$ cannot be solved for λ_1 and λ_2, or equivalently that u is not a linear combination of a_1 and a_2.

(d) Yes, it is easy to see by inspection that $v = (a_1 + a_2)/5 = 0.2a_1 + 0.2a_2$, so v is a linear combination of a_1 and a_2, or in other words $v \in \text{span}(a_1, a_2)$.

Exercise 4. Put

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \\ 1 & 2 & 3 \end{bmatrix} \qquad b_1 = \begin{bmatrix} 1 \\ 11 \\ 1 \end{bmatrix} \quad b_2 = \begin{bmatrix} 1 \\ 12 \\ 1 \end{bmatrix} \quad b_3 = \begin{bmatrix} 1 \\ 13 \\ 1 \end{bmatrix}$$

For each of the following subspaces, give an example of a nonzero vector that lies in the subspace, and an example of a nonzero vector that does not lie in the subspace.

$$\begin{aligned} V_0 &= \text{span}(b_1, b_2, b_3) \\ V_1 &= \text{ann}(b_1, b_2, b_3) \\ V_2 &= \text{img}(A) \\ V_3 &= \ker(A). \end{aligned}$$

Solution:

(a) The obvious example of a vector lying in $V_0 = \text{span}(b_1, b_2, b_3)$ is just the vector $b_1 = \begin{bmatrix} 1 & 11 & 1 \end{bmatrix}^T$ itself. Indeed, b_1 can be written as $1 \times b_1 + 0 \times b_2 + 0 \times b_3$, so it is a linear combination of b_1, b_2 and b_3, so it is an element of $\text{span}(b_1, b_2, b_3) = V_0$. On the other hand, the vector $e_1 = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}^T$ is not an element of V_0. Indeed, in each of the vectors b_i, the first and last entries are the same. Any element of V_0 is, by definition, a linear combination of the b_i and so also has the first and last entries the same. As e_1 does not have the first and last entries the same, it cannot be an element of V_0.

(b) The vector $u = \begin{bmatrix} 1 & 0 & -1 \end{bmatrix}^T$ has $u.b_1 = 1 \times 1 + 0 \times 11 + (-1) \times 1 = 0$ and similarly $u.b_2 = u.b_3 = 0$, so $u \in \text{ann}(b_1, b_2, b_3) = V_1$. On the other hand, the vector $e_1 = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}^T$ has $e_1.b_1 \neq 0$, so $e_1 \notin V_1$.

(c) The vector $v = \begin{bmatrix} 1 & 1 & 1 \end{bmatrix}^T$ can be written as $v = Ae_1$, so $u \in \text{img}(A) = V_2$. On the other hand, if y is any element of $\text{img}(A)$, then y can be written as Ax for some x, so

$$\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \\ 1 & 2 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_1 + 2x_2 + 3x_3 \\ x_1 + 2x_2 + 3x_3 \\ x_1 + 2x_2 + 3x_3 \end{bmatrix},$$

so y_1, y_2 and y_3 are all the same. In the vector e_1 the three entries are not all the same, so e_1 is not an element of $\text{img}(A)$.

(d) The vector $w = \begin{bmatrix} 2 & -1 & 0 \end{bmatrix}^T$ satisfies

$$Aw = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \\ 1 & 2 & 3 \end{bmatrix} \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

so $w \in \ker(A) = V_3$. On the other hand, we have $Ae_1 = \begin{bmatrix} 1 & 1 & 1 \end{bmatrix}^T \neq 0$, so $e_1 \notin \ker(A)$.

Exercise 5. Put

$$a_1 = \begin{bmatrix} 1 \\ 1 \\ 2 \\ 2 \end{bmatrix} \qquad a_2 = \begin{bmatrix} 2 \\ 2 \\ 1 \\ 1 \end{bmatrix} \qquad b_1 = \begin{bmatrix} 3 \\ -3 \\ 4 \\ -4 \end{bmatrix} \qquad b_2 = \begin{bmatrix} 4 \\ -4 \\ 3 \\ -3 \end{bmatrix}.$$

Show that $\text{span}(a_1, a_2) \subseteq \text{ann}(b_1, b_2)$.

Solution: First, we have

$$\begin{aligned} a_1.b_1 &= 3-3+8-8 = 0 \\ a_1.b_2 &= 4-4+6-6 = 0 \\ a_2.b_1 &= 6-6+4-4 = 0 \\ a_2.b_2 &= 8-8+3-3 = 0. \end{aligned}$$

Now consider an arbitrary element $v \in \operatorname{span}(a_1, a_2)$. By the definition of $\operatorname{span}(a_1, a_2)$, this means that v can be expressed as $v = \lambda_1 a_1 + \lambda_2 a_2$ for some scalars λ_1 and λ_2. This gives

$$\begin{aligned} v.b_1 &= (\lambda_1 a_1 + \lambda_2 a_2).b_1 = \lambda_1(a_1.b_1) + \lambda_2(a_2.b_1) = \lambda_1 \times 0 + \lambda_2 \times 0 = 0 \\ v.b_2 &= (\lambda_1 a_1 + \lambda_2 a_2).b_2 = \lambda_1(a_1.b_2) + \lambda_2(a_2.b_2) = \lambda_1 \times 0 + \lambda_2 \times 0 = 0. \end{aligned}$$

As $v.b_1 = v.b_2 = 0$, we have $v \in \operatorname{ann}(b_1, b_2)$. As this holds for every element of $\operatorname{span}(a_1, a_2)$, we have $\operatorname{span}(a_1, a_2) \subseteq \operatorname{ann}(b_1, b_2)$ as claimed.

Lecture 16

Exercise 6. Put $V = \operatorname{span}(v_1, v_2, v_3)$, where

$$\begin{aligned} v_1 &= \begin{bmatrix} 0 & 2 & 6 & 10 & 1 & 0 \end{bmatrix}^T \\ v_2 &= \begin{bmatrix} 0 & 1 & 3 & 5 & 1 & -3 \end{bmatrix}^T \\ v_3 &= \begin{bmatrix} 0 & 3 & 9 & 15 & 1 & 3 \end{bmatrix}^T. \end{aligned}$$

(a) What is the dimension of V?
(b) What is the canonical basis for V?
(c) What is the set $J(V)$ of jumps for V?

Solution: We can row-reduce the matrix $A = [v_1|v_2|v_3]^T$ as follows:

$$A = \begin{bmatrix} 0 & 2 & 6 & 10 & 1 & 0 \\ 0 & 1 & 3 & 5 & 1 & -3 \\ 0 & 3 & 9 & 15 & 1 & 3 \end{bmatrix} \to \begin{bmatrix} 0 & 0 & 0 & 0 & -1 & 6 \\ 0 & 1 & 3 & 5 & 1 & -3 \\ 0 & 0 & 0 & 0 & -2 & 12 \end{bmatrix} \to \begin{bmatrix} 0 & 1 & 3 & 5 & 0 & 3 \\ 0 & 0 & 0 & 0 & 1 & -6 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} = B.$$

According to Method 20.14, the canonical basis for V consists of the transposes of the nonzero rows in B, or in other words the vectors

$$u_1 = \begin{bmatrix} 0 & 1 & 3 & 5 & 0 & 3 \end{bmatrix}^T \qquad u_2 = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 & -6 \end{bmatrix}.$$

As this basis consists of two vectors, we have $\dim(V) = 2$. According to Lemma 20.13, the jumps for V are the pivot columns for the above matrix B. There are pivots in columns 2 and 5, so $J(V) = \{2, 5\}$.

Exercise 7. Let V be the set of all vectors of the form

$$v = \begin{bmatrix} p+q & p+2q & p+r & p+3r \end{bmatrix}^T.$$

You may assume that this is a subspace. Find a list of vectors that spans V, and then find the canonical basis for V.

Solution: A general element of V has the form

$$v = \begin{bmatrix} p+q & p+2q & p+r & p+3r \end{bmatrix} = p \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} + q \begin{bmatrix} 1 \\ 2 \\ 0 \\ 0 \end{bmatrix} + r \begin{bmatrix} 0 \\ 0 \\ 1 \\ 3 \end{bmatrix}.$$

In other words, the elements of V are precisely the linear combinations of the vectors

$$v_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \qquad v_2 = \begin{bmatrix} 1 \\ 2 \\ 0 \\ 0 \end{bmatrix} \qquad v_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 3 \end{bmatrix}.$$

For the canonical basis, we perform the following row-reduction:

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 0 & 0 \\ 0 & 0 & 1 & 3 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & -1 & -1 \\ 0 & 0 & 1 & 3 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 2 & 2 \\ 0 & 1 & -1 & -1 \\ 0 & 0 & 1 & 3 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 & -4 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 3 \end{bmatrix}.$$

We conclude that the canonical basis consists of the vectors

$$w_1 = \begin{bmatrix} 1 & 0 & 0 & -4 \end{bmatrix}^T \qquad w_2 = \begin{bmatrix} 0 & 1 & 0 & 2 \end{bmatrix}^T \qquad w_3 = \begin{bmatrix} 0 & 0 & 1 & 3 \end{bmatrix}^T.$$

Exercise 8. Put $V = \text{span}(e_1 - e_2, e_2 - e_3, \dots, e_{n-1} - e_n) \subseteq \mathbb{R}^n$, where e_i is the i'th standard basis vector for $\mathbb{R}^n$.

(a) What is the dimension of V?
(b) What is the canonical basis for V?
(c) What is the set $J(V)$ of jumps for V?

(You can start by doing the case $n = 5$ by row-reduction if you like, but ideally you should give an answer for the general case, together with a more abstract proof that your answer is correct.)

Solution: Put $v_i = e_i - e_{i+1}$, so $V = \text{span}(v_1, \dots, v_{n-1})$. For the case $n = 5$ we have can row-reduce the matrix $A = [v_1|v_2|v_3|v_4]^T$ as follows:

$$\begin{bmatrix} 1 & -1 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 1 & -1 \end{bmatrix} \to \begin{bmatrix} 1 & -1 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 1 & -1 \end{bmatrix} \to \begin{bmatrix} 1 & -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 1 & -1 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 1 & -1 \end{bmatrix}$$

The final matrix B can be described as $[w_1|w_2|w_3|w_4]^T$, where $w_i = e_i - e_4$. It follows that these vectors w_i form the canonical basis for V, so $\dim(V) = 4$. Moreover, the set of jumps for V is the set of pivot columns for B, namely $\{1, 2, 3, 4\}$.

The same pattern works for general n. In more detail, we can define vectors $w_1, \dots, w_{n-1}$ by $w_i = e_i - e_n$ and $W = \text{span}(w_1, \dots, w_{n-1})$. For $i < n - 1$ we have

$$v_i = e_i - e_{i+1} = (e_i - e_n) - (e_{i+1} - e_n) = w_i - w_{i+1},$$

whereas v_{n-1} is just equal to w_{n-1}. This shows that $v_i \in W$ for all i, and it follows that $V \subseteq W$. In the opposite direction, we have

$$v_i + v_{i+1} + \cdots + v_{n-1} = (e_i - e_{i+1}) + (e_{i+1} - e_{i+2}) + \cdots + (e_{n-1} - e_n) = e_i - e_n = w_i,$$

which shows that $w_i \in V$ for all i, and thus that $W \subseteq V$. It follows that $W = V$, so the list $\mathcal{W} = w_1, \dots, w_{n-1}$ spans V. The corresponding matrix $B = [w_1|\cdots|w_{n-1}]^T$ is clearly in RREF, so $\mathcal{W}$ is in fact the canonical basis for V. It follows that $\dim(V) = n - 1$ and $J(V) = \{1, 2, \dots, n - 1\}$.

Exercise 9. Put $V = \text{ann}(a_1, a_2, a_3) \subseteq \mathbb{R}^6$, where

$$\begin{aligned} a_1 &= \begin{bmatrix} 1 & 1 & 2 & 3 & 3 & 2 \end{bmatrix}^T \\ a_2 &= \begin{bmatrix} 3 & 3 & 2 & 1 & 1 & 2 \end{bmatrix}^T \\ a_3 &= \begin{bmatrix} 0 & 0 & 1 & 1 & 1 & 1 \end{bmatrix}^T. \end{aligned}$$

Find the canonical basis for V.

Solution: The equations $a_3.x = a_2.x = a_1.x = 0$ can be written as

$$\begin{aligned} x_6 + x_5 + x_4 + x_3 &= 0 \\ 2x_6 + x_5 + x_4 + 2x_3 + 3x_2 + x_1 &= 0 \\ 2x_6 + 3x_5 + 3x_4 + 2x_3 + x_2 + x_1 &= 0. \end{aligned}$$

The matrix A on the left below is $[a_1|a_2|a_3]^T$; the matrix A^* on the right is obtained by turning A through 180° and is the matrix of coefficients in the above system of equations.

$$A = \begin{bmatrix} 1 & 1 & 2 & 3 & 3 & 2 \\ 3 & 3 & 2 & 1 & 1 & 2 \\ 0 & 0 & 1 & 1 & 1 & 1 \end{bmatrix} \qquad A^* = \begin{bmatrix} 1 & 1 & 1 & 1 & 0 & 0 \\ 2 & 1 & 1 & 2 & 3 & 3 \\ 2 & 3 & 3 & 2 & 1 & 1 \end{bmatrix}.$$

We can row-reduce A^* as follows:

$$A^* \to \begin{bmatrix} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & -1 & -1 & 0 & 3 & 3 \\ 0 & 1 & 1 & 0 & 1 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 & 1 & 3 & 3 \\ 0 & 1 & 1 & 0 & -3 & -3 \\ 0 & 0 & 0 & 0 & 4 & 4 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 \end{bmatrix} = B^*$$

The matrix B^* corresponds to the system of equations

$$\begin{aligned} x_6 + x_3 &= 0 \\ x_5 + x_4 &= 0 \\ x_2 + x_1 &= 0, \end{aligned}$$

which can be rewritten as $x_6 = -x_3$ and $x_5 = -x_4$ and $x_2 = -x_1$. This gives

$$x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{bmatrix} = \begin{bmatrix} x_1 \\ -x_1 \\ x_3 \\ x_4 \\ -x_4 \\ -x_3 \end{bmatrix} = x_1 \begin{bmatrix} 1 \\ -1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ -1 \end{bmatrix} + x_4 \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ -1 \\ 0 \end{bmatrix}.$$

It follows that the vectors

$$\begin{aligned} v_1 &= \begin{bmatrix} 1 & -1 & 0 & 0 & 0 & 0 \end{bmatrix}^T \\ v_2 &= \begin{bmatrix} 0 & 0 & 1 & 0 & 0 & -1 \end{bmatrix}^T \\ v_3 &= \begin{bmatrix} 0 & 0 & 0 & 1 & -1 & 0 \end{bmatrix}^T \end{aligned}$$

form the canonical basis for V.

The calculation can be written more compactly in terms of Method 20.23. The matrix B^* has pivot columns 1, 2 and 5, and non-pivot columns 3, 4 and 6. Deleting the pivot columns leaves the matrix

$$C^* = \begin{bmatrix} c_1^T \\ \hline c_2^T \\ \hline c_3^T \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

We then construct the matrix

$$D^* = \begin{bmatrix} -c_1 & -c_2 & e_1 & e_2 & -c_3 & e_3 \end{bmatrix} = \begin{bmatrix} 0 & -1 & 1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 1 \end{bmatrix}$$

and rotate it to get

$$D = \begin{bmatrix} 1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & -1 \\ 0 & 0 & 0 & 1 & -1 & 0 \end{bmatrix}.$$

The canonical basis vectors v_i appear as the rows of D.

Exercise 10. Put

$$A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 2 & 1 \\ 1 & 3 & 3 & 1 \\ 1 & 4 & 4 & 1 \end{bmatrix}.$$

Find the canonical basis for $\operatorname{img}(A)$, and the canonical basis for $\ker(A)$.

Solution: First, let $a_1, \dots, a_4$ be the columns of A. Proposition 19.19 tellus us that $\operatorname{img}(A) = \operatorname{span}(a_1, \dots, a_4)$. To find the canonical basis for this space, Method 20.14 tells us that we should form the matrix whose rows are $a_1^T, \dots, a_4^T$, but that matrix is just A^T. We can row-reduce A^T as follows:

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 \\ 1 & 2 & 3 & 4 \\ 1 & 1 & 1 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & -1 & -2 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

By looking at the transposed rows of the final matrix, we see that the canonical basis for $\operatorname{img}(A)$ consists of the vectors

$$u_1 = \begin{bmatrix} 1 \\ 0 \\ -1 \\ -2 \end{bmatrix} \qquad \text{and} \qquad u_2 = \begin{bmatrix} 0 \\ 1 \\ 2 \\ 3 \end{bmatrix}.$$

Next, we recall that $\ker(A)$ is the set of vectors x that satisfy $Ax = 0$. After noting that

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 2 & 1 \\ 1 & 3 & 3 & 1 \\ 1 & 4 & 4 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} x_1 + x_2 + x_3 + x_4 \\ x_1 + 2x_2 + 2x_3 + x_4 \\ x_1 + 3x_2 + 3x_3 + x_4 \\ x_1 + 4x_2 + 4x_3 + x_4 \end{bmatrix},$$

we see that $\ker(A)$ is the set of solutions to the equations

$$\begin{aligned} x_1 + x_2 + x_3 + x_4 &= 0 \\ x_1 + 2x_2 + 2x_3 + x_4 &= 0 \\ x_1 + 3x_2 + 3x_3 + x_4 &= 0 \\ x_1 + 4x_2 + 4x_3 + x_4 &= 0. \end{aligned}$$

These are easily solved to give $x_4 = -x_1$ and $x_3 = -x_2$ with x_1 and x_2 arbitrary. (In order to get the canonical basis rather than any other basis, we need to write things this way around, with the higher-numbered variables on the left written in terms of the lower-numbered variables on the right.) This gives

$$x = \begin{bmatrix} x_1 \\ x_2 \\ -x_2 \\ -x_1 \end{bmatrix} = x_1 \begin{bmatrix} 1 \\ 0 \\ 0 \\ -1 \end{bmatrix} + x_2 \begin{bmatrix} 0 \\ 1 \\ -1 \\ 0 \end{bmatrix}.$$

From this we see that the canonical basis for $\ker(A)$ consists of the vectors

$$v_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ -1 \end{bmatrix} \qquad \text{and} \qquad v_2 = \begin{bmatrix} 0 \\ 1 \\ -1 \\ 0 \end{bmatrix}.$$