MAS201 PROBLEM SHEET 9

Lecture 17

Exercise 1. Consider the vectors

$$v_1 = \begin{bmatrix} -1 \\ 2 \\ -1 \\ 3 \end{bmatrix} \qquad v_2 = \begin{bmatrix} 1 \\ -1 \\ 2 \\ -2 \end{bmatrix} \qquad v_3 = \begin{bmatrix} 1 \\ 0 \\ 3 \\ -1 \end{bmatrix} \qquad w_1 = \begin{bmatrix} -1 \\ 5 \\ 2 \\ 6 \end{bmatrix} \qquad w_2 = \begin{bmatrix} 1 \\ 1 \\ 4 \\ 0 \end{bmatrix}$$

(a) Show that $\mathrm{span}(v_1, v_2, v_3) = \mathrm{span}(v_1, v_2) = \mathrm{span}(w_1, w_2)$.
(b) Find $\dim(\mathrm{span}(v_1, v_2, v_3, w_1, w_2))$.

Solution: We will first give a solution that involves observing various identities between the given vectors, then a longer but more systematic solution by row-reduction.

First, we observe that $v_3 = v_1 + 2v_2$. This allows us to rewrite any linear combination of v_1, v_2 and v_3 as a linear combination of v_1 and v_2 alone. Thus, we have $\mathrm{span}(v_1, v_2, v_3) = \mathrm{span}(v_1, v_2)$.

Next, we observe that $w_1 = 4v_1 + 3v_2$ and $w_2 = 2v_1 + 3v_2$. This shows that $w_1, w_2 \in \mathrm{span}(v_1, v_2)$ and so $\mathrm{span}(w_1, w_2) \subseteq \mathrm{span}(v_1, v_2)$. In the opposite direction, we have $v_1 = (w_1 - w_2)/2$ and $v_2 = (2w_2 - w_1)/3$, which shows that $v_1, v_2 \in \mathrm{span}(w_1, w_2)$ and so $\mathrm{span}(v_1, v_2) \subseteq \mathrm{span}(w_1, w_2)$.

We now see that all of the given vectors are linear combinations of v_1 and v_2, so the space $V = \mathrm{span}(v_1, v_2, v_3, w_1, w_2)$ is just the same as $\mathrm{span}(v_1, v_2)$. Recall that a list of two nonzero vectors is only linearly dependent if the vectors are scalar multiples of each other. This is clearly not the case for v_1 and v_2, so we see that the list v_1, v_2 is a basis for V, so $\dim(V) = 2$.

The more systematic approach is just to find the canonical bases for all the spaces involved. We have

$$[v_1|v_2|v_3]^T = \begin{bmatrix} -1 & 2 & -1 & 3 \\ 1 & -1 & 2 & -2 \\ 1 & 0 & 3 & -1 \end{bmatrix} \to \begin{bmatrix} 1 & -2 & 1 & -3 \\ 0 & 1 & 1 & 1 \\ 0 & 2 & 2 & 2 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 3 & -1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

It follows that the vectors $a_1 = \begin{bmatrix} 1 & 0 & 3 & -1 \end{bmatrix}^T$ and $a_2 = \begin{bmatrix} 0 & 1 & 1 & 1 \end{bmatrix}^T$ form the canonical basis for $\mathrm{span}(v_1, v_2, v_3)$. We can perform the same row-reduction leaving out the last row to see that a_1 and a_2 also form the canonical basis for $\mathrm{span}(v_1, v_2)$, so $\mathrm{span}(v_1, v_2, v_3) = \mathrm{span}(v_1, v_2)$. Similarly, we have

$$[w_1|w_2]^T = \begin{bmatrix} -1 & 5 & 2 & 6 \\ 1 & 1 & 4 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & -5 & -2 & -6 \\ 0 & 6 & 6 & 6 \end{bmatrix} \begin{bmatrix} 1 & 0 & 3 & -1 \\ 0 & 1 & 1 & 1 \end{bmatrix} = [a_1|a_2]^T$$

This shows that a_1 and a_2 also form the canonical basis for $\mathrm{span}(w_1, w_2)$, so $\mathrm{span}(v_1, v_2, v_3) = \mathrm{span}(v_1, v_2) = \mathrm{span}(w_1, w_2)$. From this it follows as before that $\mathrm{span}(v_1, v_2, v_3, w_1, w_2)$ is yet another description of the same space, and the canonical basis has two vectors so the dimension is two.

Exercise 2. Put

$$v_1 = \begin{bmatrix} 1 \\ 3 \\ 5 \\ 3 \end{bmatrix} \; v_2 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ -3 \end{bmatrix} \; w_1 = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix} \; w_2 = \begin{bmatrix} 3 \\ 2 \\ 1 \\ 0 \end{bmatrix}$$

and $V = \mathrm{span}(v_1, v_2)$ and $W = \mathrm{span}(w_1, w_2)$.

(a) Find the canonical basis for $V + W$.
(b) Find vectors a_1 and a_2 such that $V = \mathrm{ann}(a_1, a_2)$.
(c) Find vectors b_1 and b_2 such that $W = \mathrm{ann}(b_1, b_2)$.
(d) Find the canonical basis for $V \cap W$.

Solution:

(a) We can row-reduce the matrix $[v_1|v_2|w_1|w_2]^T$ as follows:

$$\begin{bmatrix} 1 & 3 & 5 & 3 \\ 1 & 1 & 1 & -3 \\ 1 & 2 & 3 & 4 \\ 3 & 2 & 1 & 0 \end{bmatrix} \to \begin{bmatrix} 0 & 2 & 4 & 6 \\ 1 & 1 & 1 & -3 \\ 0 & 1 & 2 & 7 \\ 0 & -1 & -2 & 9 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & 1 & -3 \\ 0 & 1 & 2 & 7 \\ 0 & 0 & 0 & -8 \\ 0 & 0 & 0 & 2 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

We deduce that the vectors

$$p_1 = \begin{bmatrix} 1 & 0 & -1 & 0 \end{bmatrix}^T \qquad p_2 = \begin{bmatrix} 0 & 1 & 2 & 0 \end{bmatrix}^T \qquad p_3 = \begin{bmatrix} 0 & 0 & 0 & 1 \end{bmatrix}^T$$

form the canonical basis for $V + W$.

(b) The equations $x.v_2 = x.v_1 = 0$ can be written as

$$-3x_4 + x_3 + x_2 + x_1 = 0$$
$$3x_4 + 5x_3 + 3x_2 + x_1 = 0.$$

These can be solved in the usual way to give $x_4 = x_2/9 + 2x_1/9$ and $x_3 = -2x_2/3 - x_1/3$. This in turn gives

$$x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \\ -2x_2/3 - x_1/3 \\ x_2/9 + 2x_1/9 \end{bmatrix} = x_1 \begin{bmatrix} 1 \\ 0 \\ -1/3 \\ 2/9 \end{bmatrix} + x_2 \begin{bmatrix} 0 \\ 1 \\ -2/3 \\ 1/9 \end{bmatrix}.$$

It follows that $V = \operatorname{ann}(a_1, a_2)$, where

$$a_1 = \begin{bmatrix} 1 & 0 & -1/3 & 2/9 \end{bmatrix}^T \qquad a_2 = \begin{bmatrix} 0 & 1 & -2/3 & 1/9 \end{bmatrix}^T.$$

(c) The method is the same as for part (b). The equations $x.w_2 = x.w_1 = 0$ can be written as

$$x_3 + 2x_2 + 3x_1 = 0$$
$$4x_4 + 3x_3 + 2x_2 + x_1 = 0$$

and these can be solved to give $x_4 = x_2 + 2x_1$ and $x_3 = -2x_2 - 3x_1$. This in turn gives

$$x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \\ -2x_2 - 3x_1 \\ x_2 + 2x_1 \end{bmatrix} = x_1 \begin{bmatrix} 1 \\ 0 \\ -3 \\ 2 \end{bmatrix} + x_2 \begin{bmatrix} 0 \\ 1 \\ -2 \\ 1 \end{bmatrix}.$$

It follows that $W = \operatorname{ann}(b_1, b_2)$, where

$$b_1 = \begin{bmatrix} 1 & 0 & -3 & 2 \end{bmatrix}^T \qquad b_2 = \begin{bmatrix} 0 & 1 & -2 & 1 \end{bmatrix}^T.$$

(d) Now $V \cap W = \operatorname{ann}(a_1, a_2) \cap \operatorname{ann}(b_1, b_2) = \operatorname{ann}(a_1, a_2, b_1, b_2)$. To save writing we will use the pure matrix method to calculate this. The relevant matrix A^* has rows consisting of the vectors b_2, b_1, a_2 and a_1 written backwards:

$$A^* = \begin{bmatrix} 1 & -2 & 1 & 0 \\ 2 & -3 & 0 & 1 \\ 1/9 & -2/3 & 1 & 0 \\ 2/9 & -1/3 & 0 & 1 \end{bmatrix}$$

This can be row-reduced as follows:

$$A^* \to \begin{bmatrix} 1 & -2 & 1 & 0 \\ 2 & -3 & 0 & 1 \\ 1 & -6 & 9 & 0 \\ 2 & -3 & 0 & 9 \end{bmatrix} \to \begin{bmatrix} 1 & -2 & 1 & 0 \\ 0 & 1 & -2 & 1 \\ 0 & -4 & 8 & 0 \\ 0 & 1 & -2 & 9 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & -3 & 0 \\ 0 & 1 & -2 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} = B^*$$

The matrix B^* corresponds to the system of equations $x_4 = 3x_2$ and $x_3 = 2x_2$ and $x_1 = 0$, so $x = x_2 \begin{bmatrix} 0 & 1 & 2 & 3 \end{bmatrix}$. It follows that $V \cap W$ is the set of multiples of the vector $q = \begin{bmatrix} 0 & 1 & 2 & 3 \end{bmatrix}^T$, so q on its own is the canonical basis for $V \cap W$.

Exercise 3. Put

$$U = \{x \in \mathbb{R}^3 \mid x_1 + 2x_2 + 2x_3 = 0\}$$
$$V = \{x \in \mathbb{R}^3 \mid 4x_1 - x_2 - x_3 = 0\}.$$

Find the canonical bases for U, V, $U + V$ and $U \cap V$.

Solution: First, we put $a = \begin{bmatrix}1 & 2 & 2\end{bmatrix}$ and $b = \begin{bmatrix}4 & -1 & -1\end{bmatrix}$. We have $a.x = x_1 + 2x_2 + 2x_3$, so U can be described as $U = \{x \mid x.a = 0\}$ or equivalently $U = \operatorname{ann}(a)$. Similarly, we have $V = \operatorname{ann}(b)$.

For $x \in U$ we have $x_3 = -x_1/2 - x_2$, so

$$x = \begin{bmatrix} x_1 \\ x_2 \\ -x_1/2 - x_2 \end{bmatrix} = x_1 \begin{bmatrix} 1 \\ 0 \\ -1/2 \end{bmatrix} + x_2 \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}.$$

It follows that the vectors $u_1 = \begin{bmatrix}1 & 0 & -1/2\end{bmatrix}^T$ and $u_2 = \begin{bmatrix}0 & 1 & -1\end{bmatrix}^T$ form the canonical basis for U.

Similarly, for $x \in V$ we have $x_3 = 4x_1 - x_2$ so

$$x = \begin{bmatrix} x_1 \\ x_2 \\ 4x_1 - x_2 \end{bmatrix} = x_1 \begin{bmatrix} 1 \\ 0 \\ 4 \end{bmatrix} + x_2 \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix},$$

so the vectors $v_1 = \begin{bmatrix}1 & 0 & 4\end{bmatrix}^T$ and $v_2 = \begin{bmatrix}0 & 1 & -1\end{bmatrix}^T$ form the canonical basis for V.

It now follows that $U + V = \operatorname{span}(u_1, u_2, v_1, v_2)$. However, we can omit v_2 because it is the same as u_2, so $U + V = \operatorname{span}(u_1, u_2, v_1)$. To find the canonical basis for this space we row-reduce the matrix $[u_1|u_2|v_1]^T$:

$$\begin{bmatrix} 1 & 0 & -1/2 \\ 0 & 1 & -1 \\ 1 & 0 & 4 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & -1/2 \\ 0 & 1 & -1 \\ 0 & 0 & 9/2 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & -1/2 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} e_1^T \\ \hline e_2^T \\ \hline e_3^T \end{bmatrix}$$

It follows that e_1, e_2, e_3 is the canonical basis for $U + V$ and so $U + V = \mathbb{R}^3$.

The dimension formula now gives

$$\dim(U \cap V) = \dim(U) + \dim(V) - \dim(U + V) = 2 + 2 - 3 = 1.$$

It follows that any nonzero vector in $U \cap V$ (considered as a list of length one) forms a basis for $U \cap V$. We have seen that the vector $w = \begin{bmatrix}0 & 1 & -1\end{bmatrix}^T = u_2 = v_2$ lies in both U and V, so it forms a basis for $U \cap V$. The first nonzero entry in w is one, so this is the canonical basis.

For a more direct approach, we can use the fact that

$$U \cap V = \operatorname{ann}(a) \cap \operatorname{ann}(b) = \operatorname{ann}(a, b).$$

The equations $x.b = x.a = 0$ can be written with the variables in decreasing order as

$$\begin{aligned} 2x_3 + 2x_2 + x_1 &= 0 \\ -x_3 - x_2 + 4x_1 &= 0. \end{aligned}$$

These equations can be solved to give $x_3 = -x_2$ and $x_1 = 0$, so

$$x = \begin{bmatrix} 0 \\ x_2 \\ -x_2 \end{bmatrix} = x_2 \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} = x_2 w.$$

Form this we again see that w is the canonical basis for $U \cap V$.

Exercise 4. Let V be the set of all vectors of the form

$$v = \begin{bmatrix} p + q & 2p - 2q & 3p + 3q & 4p - 4q \end{bmatrix}^T.$$

(a) Find vectors v_1 and v_2 such that $V = \operatorname{span}(v_1, v_2)$.
(b) Find vectors w_1 and w_2 such that $V = \operatorname{ann}(w_1, w_2)$.

Solution:

(a) A general element $v \in V$ can be written as

$$v = \begin{bmatrix} p + q \\ 2p - 2q \\ 3p + 3q \\ 4p - 4q \end{bmatrix} = p \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix} + q \begin{bmatrix} 1 \\ -2 \\ 3 \\ -4 \end{bmatrix}.$$

It follows that if we put $v_1 = \begin{bmatrix}1 & 2 & 3 & 4\end{bmatrix}^T$ and $v_2 = \begin{bmatrix}1 & -2 & 3 & -4\end{bmatrix}^T$ then the elements of V are precisely the linear combinations of v_1 and v_2, or in other words $V = \operatorname{span}(v_1, v_2)$.

If we want we can tidy this up by row-reduction:

$$[v_1|v_2]^T = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 1 & -2 & 3 & -4 \end{bmatrix} \to \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & -4 & 0 & -8 \end{bmatrix} \to \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & 0 & 2 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 3 & 0 \\ 0 & 1 & 0 & 2 \end{bmatrix}.$$

It follows that V can also be described as $\text{span}(v_1', v_2')$, where $v_1' = \begin{bmatrix} 1 & 0 & 3 & 0 \end{bmatrix}^T$ and $v_2' = \begin{bmatrix} 0 & 1 & 0 & 2 \end{bmatrix}^T$. (In fact, v_1' and v_2' form the canonical basis for V.)

(b) The equations $x.v_2 = 0$ and $x.v_1 = 0$ can be written as

$$\begin{aligned} -4x_4 + 3x_3 - 2x_2 + x_1 &= 0 \\ 4x_4 + 3x_3 + 2x_2 + x_1 &= 0. \end{aligned}$$

By adding the above equations we get $6x_2 + 2x_1 = 0$ or $x_3 = -x_1/3$. By subtracting the above equations we get $8x_4 + 4x_2 = 0$ or $x_4 = -x_2/2$. This gives

$$x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \\ -x_1/3 \\ -x_2/2 \end{bmatrix} = x_1 \begin{bmatrix} 1 \\ 0 \\ -1/3 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} 0 \\ 1 \\ 0 \\ -1/2 \end{bmatrix}.$$

It follows that $V = \text{ann}(w_1, w_2)$, where $w_1 = \begin{bmatrix} 1 & 0 & -1/3 & 0 \end{bmatrix}^T$ and $w_2 = \begin{bmatrix} 0 & 1 & 0 & -1/2 \end{bmatrix}$.

Note that we could also have started with the equations $x.v_2' = x.v_1' = 0$ instead of $x.v_2 = x.v_1 = 0$ and we would still have obtained the same vectors w_i.

Exercise 5. For each of the following configurations, either find an example, or show that no example is possible.

(a) Subspaces $U, V \leq \mathbb{R}^4$ with $\dim(U) = \dim(V) = 3$ and $\dim(U \cap V) = 1$.
(b) Subspaces $U, V \leq \mathbb{R}^4$ with $\dim(U) = \dim(V) = 3$ and $\dim(U \cap V) = 2$.
(c) Subspaces $U, V \leq \mathbb{R}^5$ with $\dim(U) = \dim(V) = 2$ and $\dim(U + V) = 5$.
(d) Subspaces $U, V \leq \mathbb{R}^3$ with $\dim(U) = \dim(V) = \dim(U + V) = \dim(U \cap V)$.

Solution: We will repeatedly use the dimension formula

$$\dim(U) + \dim(V) = \dim(U + V) + \dim(U \cap V).$$

(a) This is not possible. Indeed, the dimension formula can be rearranged to give $\dim(U + V) = \dim(U) + \dim(V) - \dim(U \cap V) = 3 + 3 - 1 = 5$, but $U + V$ is a subspace of $\mathbb{R}^4$, so it cannot have dimension greater than 4.

(b) The simplest example is

$$\begin{aligned} U &= \text{span}(e_1, e_2, e_3) = \{\begin{bmatrix} w & x & y & 0 \end{bmatrix}^T \mid w, x, y \in \mathbb{R}\} \\ V &= \text{span}(e_1, e_2, e_4) = \{\begin{bmatrix} w & x & 0 & z \end{bmatrix}^T \mid w, x, z \in \mathbb{R}\} \\ U \cap V &= \text{span}(e_1, e_2) = \{\begin{bmatrix} w & x & 0 & 0 \end{bmatrix}^T \mid w, x \in \mathbb{R}\}. \end{aligned}$$

(c) This is not possible. Indeed, the dimension formula can be rearranged to give $\dim(U \cap V) = \dim(U) + \dim(V) - \dim(U + V) = 2 + 2 - 5 = -1$, but no subspace can have negative dimension.

(d) The minimal example here is to take $U = V = \{0\}$, so $U + V = U \cap V = \{0\}$ and $\dim(U) = \dim(V) = \dim(U + V) = \dim(U \cap V) = 0$. More generally, we can choose U to be any subspace of $\mathbb{R}^3$ (of dimension d, say) and take $V = U$. We then have $U + V = U + U = U$ and $U \cap V = U \cap U = U$ so $\dim(U) = \dim(V) = \dim(U + V) = \dim(U \cap V) = d$.

Lecture 18

Exercise 6. Find the ranks of the following matrices:

$$A = \begin{bmatrix} 0 & 1 & 2 \\ -1 & 0 & 3 \\ -2 & -3 & 0 \end{bmatrix} \quad B = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 2 & 3 & 4 & 5 & 6 & 7 \\ 3 & 4 & 5 & 6 & 7 & 8 \end{bmatrix} \quad C = \begin{bmatrix} 1 & 10 & 100 \\ 10 & 100 & 1000 \\ 100 & 1000 & 10000 \end{bmatrix} \quad D = \begin{bmatrix} 1 & 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 \end{bmatrix}$$

Solution: The rank of a matrix M is the number of nonzero rows in the row-reduced form of M. We have row-reductions as follows:

$$A = \begin{bmatrix} 0 & 1 & 2 \\ -1 & 0 & 3 \\ -2 & -3 & 0 \end{bmatrix} \to \begin{bmatrix} 0 & 1 & 2 \\ 1 & 0 & -3 \\ -2 & -3 & 0 \end{bmatrix} \to \begin{bmatrix} 0 & 1 & 2 \\ 1 & 0 & -3 \\ 0 & -3 & -6 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & -3 \\ 0 & 1 & 2 \\ 0 & -3 & -6 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & -3 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}$$

$$B = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 2 & 3 & 4 & 5 & 6 & 7 \\ 3 & 4 & 5 & 6 & 7 & 8 \end{bmatrix} \to \begin{bmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 0 & -1 & -2 & -3 & -4 & -5 \\ 0 & -2 & -4 & -6 & -8 & -10 \end{bmatrix} \to$$

$$\begin{bmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 0 & 1 & 2 & 3 & 4 & 5 \\ 0 & -2 & -4 & -6 & -8 & -10 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & -1 & -2 & -3 & -4 \\ 0 & 1 & 2 & 3 & 4 & 5 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$C = \begin{bmatrix} 1 & 10 & 100 \\ 10 & 100 & 1000 \\ 100 & 1000 & 10000 \end{bmatrix} \to \begin{bmatrix} 1 & 10 & 100 \\ 0 & 0 & 0 \\ 100 & 1000 & 10000 \end{bmatrix} \to \begin{bmatrix} 1 & 10 & 100 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$D = \begin{bmatrix} 1 & 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 1 & 1 & 1 \\ 0 & 0 & -1 & -1 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 1 & 1 & 1 \\ 0 & 0 & -1 & -1 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

From this we see that $\text{rank}(A) = \text{rank}(B) = 2$ and $\text{rank}(C) = 1$ and $\text{rank}(D) = 3$.

Exercise 7. Give examples as follows, or explain why no such examples are possible.

(a) A 3×5 matrix of rank 4.
(b) A 3×3 matrix of rank 1, in which none of the entries are zero.
(c) A 2×4 matrix A such that A has rank 1 and A^T has rank 2.
(d) A 3×3 matrix A such that $A + A^T = 0$ and A has rank 2.
(e) An invertible 3×3 matrix of rank 2.
(f) A matrix in RREF with rank 1 and 4 nonzero columns.

Solution:

(a) This is not possible, because the rank of any $m \times n$ matrix is at most the minimum of n and m, so a 3×5 matrix cannot have rank larger than 3.
(b) The simplest example is $A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$.
(c) This is not possible, because A and A^T always have the same rank.
(d) The simplest example is $A = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix}$.
(e) This is not possible. If A is an *invertible* $n \times n$ matrix, then the columns form a basis for $\mathbb{R}^n$, which means that the rank must be n.
(f) One example is the matrix $\begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$.

Exercise 8. Consider the following matrices, which depend on a parameter t.

$$A = \begin{bmatrix} 1 & 0 \\ 0 & (t-3)(t-4) \end{bmatrix} \qquad B = \begin{bmatrix} 1 & t \\ t & 2t-1 \end{bmatrix} \qquad C = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & t \\ 1 & 4 & t^2 \end{bmatrix} \qquad D = \begin{bmatrix} 1 & 1 & 1 & 1 & 0 \\ 1 & 2 & t & 3 & t \\ 1 & 4 & t^2 & 7 & 3 \end{bmatrix}$$

It should be clear that A usually has rank two, except that when $t = 3$ or $t = 4$ the second row becomes zero and so the rank is only one. In the same way, for each of the other matrices, there is a usual value for the rank, but the rank drops for some exceptional values of t.

(1) Simplify B by row and column operations. Do not divide any row or column by anything that depends on t, but make B as simple as you can without such divisions.
(2) What is the usual rank of B?
(3) What is the exceptional value of t for which the rank of B is lower? What is the rank in that case?

(4) What is the usual rank of C, and what are the exceptional cases? (Use the same method as for B.)
(5) What is the usual rank of D, and what are the exceptional cases? (**Hint:** how is D related to C?)

Solution:

(1) Subtract t times the first row from the second row, then subtract t times the first column from the second column:

$$B = \begin{bmatrix} 1 & t \\ t & 2t-1 \end{bmatrix} \to \begin{bmatrix} 1 & t \\ 0 & -t^2+2t-1 \end{bmatrix} \to \begin{bmatrix} 1 & 0 \\ 0 & -t^2+2t-1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & -(t-1)^2 \end{bmatrix} = B'.$$

We might now be tempted to divide the second row by $-(t-1)^2$ to get the identity matrix. However, that would not be valid when $t = 1$, because then we would be dividing by zero. It is for this reason that the question tells you not to divide by anyhing that depends on t.

(2) As row and column operations do not affect the rank, we have $\mathrm{rank}(B) = \mathrm{rank}(B')$. If $t \neq 1$ then it is clear that the two rows in B' are linearly independent and so $\mathrm{rank}(B) = \mathrm{rank}(B') = 2$; this is the usual case.

(3) In the exceptional case where $t = 1$ we have $B' = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ and it is clear that $\mathrm{rank}(B) = \mathrm{rank}(B') = 1$.

(4) We can simplify C by row and column operations as follows.

$$C = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & t \\ 1 & 4 & t^2 \end{bmatrix} \xrightarrow{1} \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & t-1 \\ 0 & 3 & t^2-1 \end{bmatrix} \xrightarrow{2} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & t-1 \\ 0 & 3 & t^2-1 \end{bmatrix} \xrightarrow{3} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 3 & t^2-3t+2 \end{bmatrix} \xrightarrow{4} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & t^2-3t+2 \end{bmatrix} = C'$$

(Step 1: subtract row 1 from the other two rows; Step 2: subtract column 1 from the other two columns; Step 3: add $1-t$ times column 2 to column 3; Step 4: subtract 3 times row 2 from row 3.) Note also that $t^2 - 3t + 2 = (t-1)(t-2)$. For most values of t this will be nonzero, so $\mathrm{rank}(C) = \mathrm{rank}(C') = 3$. The exceptional cases are where $t = 1$ or $t = 2$, in which case $C' = \left[\begin{smallmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{smallmatrix}\right]$ and $\mathrm{rank}(C) = \mathrm{rank}(C') = 2$.

(5) C consists of the first three columns of D. If $t \neq 1, 2$ then $\mathrm{rank}(C) = 3$ so the columns of C span $\mathbb{R}^3$, so the columns of D certainly span $\mathbb{R}^3$, so $\mathrm{rank}(D) = 3$. In the case $t = 1$ we can write down D and simplify by column operations as follows:

$$D = \begin{bmatrix} 1 & 1 & 1 & 1 & 0 \\ 1 & 2 & 1 & 3 & 1 \\ 1 & 4 & 1 & 7 & 3 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 2 & 1 \\ 1 & 3 & 0 & 6 & 3 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ -2 & 3 & 0 & 0 & 0 \end{bmatrix} = D'.$$

It is clear that in this case we have $\mathrm{rank}(D) = \mathrm{rank}(D') = 2$. In the other exceptional case where $t = 2$ we can write down D and simplify by column operations as follows:

$$D = \begin{bmatrix} 1 & 1 & 1 & 1 & 0 \\ 1 & 2 & 2 & 3 & 2 \\ 1 & 4 & 4 & 7 & 3 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 2 & 2 \\ 1 & 3 & 0 & 6 & 3 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 1 & 3 & 0 & 0 & -3 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{bmatrix} = D''.$$

It is clear from this that the case $t = 2$ is not in fact exceptional for D, because we have $\mathrm{rank}(D) = \mathrm{rank}(D'') = 3$ in that case (which is the same answer as for every other value of t except $t = 1$).