MAS201 PROBLEM SHEET 10

Exercise 1. Show that if A is an orthogonal matrix then $\det(A) = \pm 1$.

Solution: By the definition of an orthogonal matrix we have $A^TA = I$, so $\det(A^T)\det(A) = \det(I) = 1$. For any square matrix A we have $\det(A^T) = \det(A)$, so we now see that $\det(A)^2 = 1$, which gives $\det(A) = \pm 1$.

Exercise 2. Consider the matrix

$$A = \frac{1}{25}\begin{bmatrix} 15 & -16 & 12 \\ 20 & 12 & -9 \\ 0 & 15 & 20 \end{bmatrix}.$$

(a) Show that A is an orthogonal matrix.
(b) Check directly that the columns of A form a basis for $\mathbb{R}^3$.
(c) Find the determinant of A.

Solution:

(a) We must show that A^TA is the identity matrix:

$$A^TA = \frac{1}{25^2}\begin{bmatrix} 15 & 20 & 0 \\ -16 & 12 & 15 \\ 12 & -9 & 20 \end{bmatrix}\begin{bmatrix} 15 & -16 & 12 \\ 20 & 12 & -9 \\ 0 & 15 & 20 \end{bmatrix}$$

$$= \frac{1}{625}\begin{bmatrix} 225+400 & -240+240 & 180-180 \\ -240+240 & 256+144+225 & -192-108+400 \\ 180-180 & -192-108+400 & 144+81+400 \end{bmatrix} = \frac{1}{625}\begin{bmatrix} 625 & 0 & 0 \\ 0 & 625 & 0 \\ 0 & 0 & 625 \end{bmatrix} = I_4.$$

(b) Let the columns of A be v_1, v_2 and v_3. The standard method is to form the matrix B whose rows are v_1^T, v_2^T and v_3^T, and see whether it row-reduces to I_3; if so, then the list v_1, v_2, v_3 is a basis. Here of course B is just A^T, and as our first step of row-reduction we will multiply all rows by 25. The remaining steps are as follows:

$$\begin{bmatrix} 15 & 20 & 0 \\ -16 & 12 & 15 \\ 12 & -9 & 20 \end{bmatrix} \to \begin{bmatrix} 1 & 4/3 & 0 \\ -16 & 12 & 15 \\ 12 & -9 & 20 \end{bmatrix} \to \begin{bmatrix} 1 & 4/3 & 0 \\ 0 & 100/3 & 15 \\ 0 & -25 & 20 \end{bmatrix} \to \begin{bmatrix} 1 & 4/3 & 0 \\ 0 & 1 & 9/20 \\ 0 & -25 & 20 \end{bmatrix} \to$$

$$\begin{bmatrix} 1 & 4/3 & 0 \\ 0 & 1 & 9/20 \\ 0 & 0 & 125/4 \end{bmatrix} \to \begin{bmatrix} 1 & 4/3 & 0 \\ 0 & 1 & 9/20 \\ 0 & 0 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 4/3 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

(c) Note that Exercise 1 tells us that $\det(A) = \pm 1$, but there is no easy shortcut to see whether it is $+1$ or -1.

One approach is by row-reduction (as in Method 12.9). In part (b) we started by multiplying all rows by 25, which gives a factor of 25^3. We never swapped any rows, by at various stages we multipled rows by $1/15$, $3/100$ and $4/125$. This gives an overall factor of

$$\mu = 25^3 \times \frac{1}{15} \times \frac{3}{100} \times \frac{4}{125} = 1.$$

The final matrix is I_3, so we conclude that $\det(A^T) = \det(I_3)/\mu = 1$. It is also a general fact that $\det(A) = \det(A^T)$, so we have $\det(A) = 1$.

An alternative is just to expand out the determinant, say down the first column:

$$\det(A) = \frac{1}{25^3}\det\begin{bmatrix} 15 & -16 & 12 \\ 20 & 12 & -9 \\ 0 & 15 & 20 \end{bmatrix}$$

$$= \frac{1}{25^3}\left(15\det\begin{bmatrix} 12 & -9 \\ 15 & 20 \end{bmatrix} - 20\det\begin{bmatrix} -16 & 12 \\ 15 & 20 \end{bmatrix}\right) = \frac{15 \times 375 - 20 \times (-500)}{15625} = 1.$$

We could instead expand along the top row:

$$\begin{aligned}
\det(A) &= \frac{1}{25^3}\det\begin{bmatrix} 15 & -16 & 12 \\ 20 & 12 & -9 \\ 0 & 15 & 20 \end{bmatrix} \\
&= \frac{1}{25^3}\left(15\det\begin{bmatrix} 12 & -9 \\ 15 & 20 \end{bmatrix} - (-16)\det\begin{bmatrix} 20 & -9 \\ 0 & 20 \end{bmatrix} + 12\det\begin{bmatrix} 20 & 12 \\ 0 & 15 \end{bmatrix}\right) \\
&= \frac{15\times 375 + 16\times 400 + 12\times 300}{15625} = 1.
\end{aligned}$$

Exercise 3. Show that the following matrices are both orthogonal.

$$A = \frac{1}{1+t^2}\begin{bmatrix} 1-t^2 & -2t \\ 2t & 1-t^2 \end{bmatrix} \qquad B = \begin{bmatrix} \sin(\phi)\cos(\theta) & \cos(\phi)\cos(\theta) & -\sin(\theta) \\ \sin(\phi)\sin(\theta) & \cos(\phi)\sin(\theta) & \cos(\theta) \\ \cos(\phi) & -\sin(\phi) & 0 \end{bmatrix}$$

Solution: First, we have

$$\begin{aligned}
A^TA &= \frac{1}{(1+t^2)^2}\begin{bmatrix} 1-t^2 & 2t \\ -2t & 1-t^2 \end{bmatrix}\begin{bmatrix} 1-t^2 & -2t \\ 2t & 1-t^2 \end{bmatrix} = \frac{1}{(1+t^2)^2}\begin{bmatrix} (1-t^2)^2+4t^2 & 0 \\ 0 & (1-t^2)^2+4t^2 \end{bmatrix} \\
&= \frac{1}{1+2t^2+t^4}\begin{bmatrix} 1+2t^2+t^4 & 0 \\ 0 & 1+2t^2+t^4 \end{bmatrix} = I_2.
\end{aligned}$$

For B, we can save writing by putting

$$x = \cos(\theta) \qquad y = \sin(\theta) \qquad u = \cos(\phi) \qquad v = \sin(\phi)$$

so that $x^2 + y^2 = u^2 + v^2 = 1$. This gives

$$\begin{aligned}
B^TB &= \begin{bmatrix} vx & vy & u \\ ux & uy & -v \\ y & -x & 0 \end{bmatrix}\begin{bmatrix} vx & ux & y \\ vy & uy & -x \\ u & -v & 0 \end{bmatrix} = \begin{bmatrix} v^2x^2+v^2y^2+u^2 & uvx^2+uvy^2-uv & vxy-vxy \\ uvx^2+uvy^2-uv & u^2x^2+u^2y^2+v^2 & uvy-uvy \\ vxy-vxy & uxy-uxy & x^2+y^2 \end{bmatrix} \\
&= \begin{bmatrix} v^2+u^2 & uv-uv & 0 \\ uv-uv & u^2+v^2 & 0 \\ 0 & 0 & x^2+y^2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}
\end{aligned}$$

Exercise 4. For each of the following, give either an example, or a proof that no example is possible.

(a) An orthonormal list of length 4 in $\mathbb{R}^3$.
(b) A 3×3 matrix that is both symmetric and orthogonal.
(c) A 4×4 matrix that is both antisymmetric and orthogonal.
(d) A 3×3 matrix that is both symmetric and antisymmetric.
(e) A 4×4 orthogonal matrix of rank 3.
(f) A 3×3 matrix that is both antisymmetric and orthogonal (think about determinants).

Solution:

(a) This is not possible. Any orthonormal list is linearly independent (by the proof of Proposition 23.4), so an orthonormal list in $\mathbb{R}^3$ cannot have length greater than 3.
(b) The obvious example is the identity matrix I_3.
(c) One example is the matrix

$$A = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{bmatrix}.$$

(d) The only possible example is the zero matrix. Indeed, if A is both symmetric and antisymmetric then $A = A^T$ and $A = -A^T$; adding these equations gives $2A = 0$ so $A = 0$.
(e) This is not possible. If A is orthogonal then the columns form an (orthonormal) basis for $\mathbb{R}^4$, so the span of the columns is all of $\mathbb{R}^4$. The rank is the dimension of the span of the columns, so it must be equal to 4.

(f) This is not possible. Recall that for any $d \times d$ matrix M we have $\det(M^T) = \det(M)$ and $\det(tM) = t^d \det M$. If A is a 3×3 antisymmetric matrix, it follows that $A^T = -A$ and so

$$\det(A) = \det(A^T) = \det(-A) = (-1)^3 \det(A) = -\det(A).$$

This means that $\det(A) = 0$, so A is not invertible. On the other hand, any orthogonal matrix A is invertible, with $A^{-1} = A^T$. It follows that a 3×3 matrix cannot be both antisymmetric and orthogonal.

Exercise 5. There are precisely eight different 3×3 matrices that are both upper triangular and orthogonal. Find them all.

Solution: The eight matrices in question are those of the form $\begin{bmatrix} \pm 1 & 0 & 0 \\ 0 & \pm 1 & 0 \\ 0 & 0 & \pm 1 \end{bmatrix}$. (The signs of the three diagonal entries can be chosen independently, so there are $2 \times 2 \times 2 = 8$ possible choices.) To see this, note that any upper triangular matrix has the form

$$A = \begin{bmatrix} a & b & c \\ 0 & d & e \\ 0 & 0 & f \end{bmatrix}.$$

If A is also orthogonal, we must have $A^T A = I_3$, or more explicitly

$$A^T A = \begin{bmatrix} a & 0 & 0 \\ b & d & 0 \\ c & e & f \end{bmatrix} \begin{bmatrix} a & b & c \\ 0 & d & e \\ 0 & 0 & f \end{bmatrix} = \begin{bmatrix} a^2 & ab & ac \\ ab & b^2 + d^2 & bc + de \\ ac & bc + de & c^2 + e^2 + f^2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Looking in the top left corner, we see that $a^2 = 1$, so $a = \pm 1$. Looking at the rest of the top row, we have $ab = ac = 0$. AS $a = \pm 1$ we can divide by a and conclude that $b = c = 0$. We now look in the middle entry of the matrix to see that $b^2 + d^2 = 1$, but $b = 0$ so $d^2 = 1$ so $d = \pm 1$. Looking at the last entry in the middle row, we see that $bc + de = 0$, but $b = c = 0$ and $d = \pm 1$ so $e = 0$. The bottom right entry now gives $f^2 = 1$ so $f = \pm 1$. In conclusion, we have

$$A = \begin{bmatrix} a & b & c \\ 0 & d & e \\ 0 & 0 & f \end{bmatrix} = \begin{bmatrix} a & 0 & 0 \\ 0 & d & 0 \\ 0 & 0 & f \end{bmatrix} = \begin{bmatrix} \pm 1 & 0 & 0 \\ 0 & \pm 1 & 0 \\ 0 & 0 & \pm 1 \end{bmatrix}$$

as claimed.

Exercise 6. Find an orthogonal diagonalisation for the following symmetric matrix:

$$A = \begin{bmatrix} 7 & 0 & 4 \\ 0 & -7 & -4 \\ 4 & -4 & 0 \end{bmatrix}.$$

Solution: The characteristic polynomial is

$$\chi_A(t) = \det \begin{bmatrix} 7-t & 0 & 4 \\ 0 & -7-t & -4 \\ 4 & -4 & -t \end{bmatrix} = (7-t)\det \begin{bmatrix} -7-t & -4 \\ -4 & -t \end{bmatrix} + 4 \det \begin{bmatrix} 0 & -7-t \\ 4 & -4 \end{bmatrix}$$

$$\det \begin{bmatrix} -7-t & -4 \\ -4 & -t \end{bmatrix} = t^2 + 7t - 16$$

$$\det\det \begin{bmatrix} 0 & -7-t \\ 4 & -4 \end{bmatrix} = 28 + 4t$$

$$\chi_A(t) = (7-t)(t^2 + 7t - 16) + 4(28 + 4t) = -t^3 + 81t = -t(t-9)(t+9).$$

It follows that the eigenvalues of A are $\lambda_1 = 0$, $\lambda_2 = 9$ and $\lambda_3 = -9$. To find the corresponding eigenvectors, we perform the following row-reductions:

$$
\begin{aligned}
A = \begin{bmatrix} 7 & 0 & 4 \\ 0 & -7 & -4 \\ 4 & -4 & 0 \end{bmatrix} &\to \begin{bmatrix} 7 & 0 & 4 \\ 0 & -7 & -4 \\ 1 & -1 & 0 \end{bmatrix} \to \begin{bmatrix} 0 & 7 & 4 \\ 0 & -7 & -4 \\ 1 & -1 & 0 \end{bmatrix} \\
&\to \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & 4/7 \\ 0 & 0 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 4/7 \\ 0 & 1 & 4/7 \\ 0 & 0 & 0 \end{bmatrix} \\
A - 9I = \begin{bmatrix} -2 & 0 & 4 \\ 0 & -16 & -4 \\ 4 & -4 & -9 \end{bmatrix} &\to \begin{bmatrix} 1 & 0 & -2 \\ 0 & -16 & -4 \\ 0 & -4 & -1 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & -1/4 \\ 0 & 0 & 0 \end{bmatrix} \\
A + 9I = \begin{bmatrix} 16 & 0 & 4 \\ 0 & 2 & -4 \\ 4 & -4 & 9 \end{bmatrix} &\to \begin{bmatrix} 16 & 0 & 4 \\ 0 & 1 & -2 \\ 4 & 0 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 1/4 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \end{bmatrix}.
\end{aligned}
$$

From the first row-reduction we see that the vector $v_1 = \begin{bmatrix} 4 & 4 & -7 \end{bmatrix}^T$ is an eigenvector of eigenvalue $\lambda_1 = 0$. From the second row-reduction we see that the vector $v_2 = \begin{bmatrix} 8 & 1 & 4 \end{bmatrix}^T$ is an eigenvector of eigenvalue $\lambda_2 = 9$. From the third row-reduction we see that the vector $v_3 = \begin{bmatrix} 1 & -8 & -4 \end{bmatrix}^T$ is an eigenvector of eigenvalue $\lambda_3 = -9$. These are automatically orthogonal, because A is symmetric and the eigenvalues are different. However, they are not orthonormal, because we have

$$
\begin{aligned}
v_1.v_1 &= 4^2 + 4^2 + (-7)^2 = 81 \\
v_2.v_2 &= 8^2 + 1^2 + 4^2 = 81 \\
v_3.v_3 &= 1^2 + (-8)^2 + (-4)^2 = 81.
\end{aligned}
$$

However, if we put $u_i = v_i/\sqrt{v_i.v_i} = v_i/9$, then the vectors u_1, u_2 and u_3 form an orthonormal basis for $\mathbb{R}^3$ consisting of eigenvectors for A. Thus, if we put

$$
\begin{aligned}
U = [u_1|u_2|u_3] &= \frac{1}{9}\begin{bmatrix} 4 & 8 & 1 \\ 4 & 1 & -8 \\ -7 & 4{}^{-}4 & \end{bmatrix} \\
D = \operatorname{diag}(\lambda_1, \lambda_2, \lambda_3) &= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 9 & 0 \\ 0 & 0 & -9 \end{bmatrix},
\end{aligned}
$$

then we have $A = UDU^{-1} = UDU^T$.

Exercise 7. Consider the matrix

$$
A = \begin{bmatrix} 1111 & 1089 & 909 & 891 \\ 1089 & 1111 & 891 & 909 \\ 909 & 891 & 1111 & 1089 \\ 891 & 909 & 1089 & 1111 \end{bmatrix}.
$$

Find a diagonalisation $A = UDU^{-1}$, where U is an orthogonal matrix.

Hint: You may assume that there are some eigenvectors of the form $\begin{bmatrix} \pm 1 & \pm 1 & \pm 1 & \pm 1 \end{bmatrix}^T$. As you work through the calculation, you should think carefully at each step about whether the general theory gives you any information that you can use before proceeding further. You should not blindly follow the standard method.

Solution: Let S be the set of vectors of the form $\begin{bmatrix} \pm 1 & \pm 1 & \pm 1 & \pm 1 \end{bmatrix}^T$. This contains $2 \times 2 \times 2 \times 2 = 16$ different vectors, and we are told that some of them are eigenvectors for A. We could just check all of them to see whether they satisfy $As = \lambda s$ for some λ. We can streamline the process a little, because s is an eigenvector if and only if $-s$ is an eigenvector, so we only need to check 8 of the possibilities. The obvious one to start with is $v_1 = \begin{bmatrix} 1 & 1 & 1 & 1 \end{bmatrix}^T$. This satisfies

$$
Av_1 = \begin{bmatrix} 1111 & 1089 & 909 & 891 \\ 1089 & 1111 & 891 & 909 \\ 909 & 891 & 1111 & 1089 \\ 891 & 909 & 1089 & 1111 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 4000 \\ 4000 \\ 4000 \\ 4000 \end{bmatrix} = 4000 v_1,
$$

so v_1 is an eigenvector of eigenvalue $\lambda_1 = 4000$. Now recall that most 4×4 symmetric matrices have four distinct eigenvalues, and that eigenvectors for distinct eigenvalues are orthogonal. This makes it natural to concentrate on the vectors in S that are orthogonal to v_1. Any such vector must contain two $+1$'s and two -1's. To specify such a vector we must choose which two of the four positions contains -1, so the number of possibilities is the binomial coefficient $\binom{4}{2} = 6$. After recalling again that s is an eigenvector if and only if $-s$ is an eigenvector, we need only check three of the six possibilities, namely

$$v_2 = \begin{bmatrix} 1 \\ 1 \\ -1 \\ -1 \end{bmatrix} \qquad v_3 = \begin{bmatrix} 1 \\ -1 \\ 1 \\ -1 \end{bmatrix} \qquad v_4 = \begin{bmatrix} 1 \\ -1 \\ -1 \\ 1 \end{bmatrix}.$$

We find that

$$Av_2 = \begin{bmatrix} 1111 & 1089 & 909 & 891 \\ 1089 & 1111 & 891 & 909 \\ 909 & 891 & 1111 & 1089 \\ 891 & 909 & 1089 & 1111 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ -1 \\ -1 \end{bmatrix} = \begin{bmatrix} 400 \\ 400 \\ -400 \\ -400 \end{bmatrix} = 400v_2$$

$$Av_3 = \begin{bmatrix} 1111 & 1089 & 909 & 891 \\ 1089 & 1111 & 891 & 909 \\ 909 & 891 & 1111 & 1089 \\ 891 & 909 & 1089 & 1111 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \\ 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 40 \\ -40 \\ 40 \\ -40 \end{bmatrix} = 40v_3$$

$$Av_4 = \begin{bmatrix} 1111 & 1089 & 909 & 891 \\ 1089 & 1111 & 891 & 909 \\ 909 & 891 & 1111 & 1089 \\ 891 & 909 & 1089 & 1111 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \\ -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 4 \\ -4 \\ -4 \\ 4 \end{bmatrix} = 4v_4.$$

We conclude that all three of these are eigenvectors, with eigenvalues $\lambda_2 = 400$, $\lambda_3 = 40$ and $\lambda_4 = 4$. We now have four eigenvectors for A with different eigenvalues, so they must all be orthogonal to each other (which is easy to see drectly) and they form a basis for $\mathbb{R}^4$. It is not an orthonormal basis, because

$$v_1.v_1 = v_2.v_2 = v_3.v_3 = v_4.v_4 = 4 \neq 1.$$

However, if we put $u_i = v_i/2$ we find that $u_i.u_i = 1$, so the list $u_1, \ldots, u_4$ is an orthonormal basis for $\mathbb{R}^4$. It follows that the matrix

$$U = [u_1|u_2|u_3|u_4] = \frac{1}{2} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & 1 & -1 \\ 1 & -1 & -1 & 1 \end{bmatrix}$$

is orthogonal. Thus, if we put

$$D = \begin{bmatrix} \lambda_1 & 0 & 0 & 0 \\ 0 & \lambda_2 & 0 & 0 \\ 0 & 0 & \lambda_3 & 0 \\ 0 & 0 & 0 & \lambda_4 \end{bmatrix} = \begin{bmatrix} 4000 & 0 & 0 & 0 \\ 0 & 400 & 0 & 0 \\ 0 & 0 & 40 & 0 \\ 0 & 0 & 0 & 4 \end{bmatrix},$$

we have an orthogonal diagonalisation $A = UDU^{-1} = UDU^T$.

Exercise 8. Diagonalise the quadratic form $Q = ax^2 + 2bxy + ay^2$ (where a and b are constants with $b > a > 0$).

Solution: This corresponds to the symmetric matrix $A = \begin{bmatrix} a & b \\ b & a \end{bmatrix}$, which has

$$\chi_A(t) = \det \begin{bmatrix} a-t & b \\ b & a-t \end{bmatrix} = t^2 - 2at + a^2 - b^2.$$

The quadratic formula gives the roots:

$$\lambda = \frac{2a \pm \sqrt{4a^2 - 4a^2 + 4b^2}}{2} = a \pm b.$$

We put $\lambda_1 = a + b > 0$ and $\lambda_2 = a - b < 0$. This gives

$$A - \lambda_1 I = \begin{bmatrix} -b & b \\ b & -b \end{bmatrix} = b \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix}$$
$$A - \lambda_2 I = \begin{bmatrix} b & b \\ b & b \end{bmatrix} = b \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}.$$

From this it is easy to see that the vectors $u_1 = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $u_2 = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ give an orthonormal basis for $\mathbb{R}^2$ consisting of eigenvectors for A. Now put

$$L_1 = \sqrt{\lambda_1} u_1 . \begin{bmatrix} x \\ y \end{bmatrix} = \sqrt{\frac{a+b}{2}} (x + y)$$
$$L_2 = \sqrt{|\lambda_1|} u_1 . \begin{bmatrix} x \\ y \end{bmatrix} = \sqrt{\frac{b-a}{2}} (x - y).$$

The general theory then tells us that $Q = L_1^2 - L_2^2$.

Exercise 9. Diagonalise the quadratic form $Q = 12wx + 10xy + 12yz$, and thus determine the rank and signature.

Solution: The corresponding matrix is

$$A = \begin{bmatrix} 0 & 6 & 0 & 0 \\ 6 & 0 & 5 & 0 \\ 0 & 5 & 0 & 6 \\ 0 & 0 & 6 & 0 \end{bmatrix}.$$

We can find the characteristic polynomial as follows:

$$\chi_A(t) = \det \begin{bmatrix} -t & 6 & 0 & 0 \\ 6 & -t & 5 & 0 \\ 0 & 5 & -t & 6 \\ 0 & 0 & 6 & -t \end{bmatrix} = -t \det \begin{bmatrix} -t & 5 & 0 \\ 5 & -t & 6 \\ 0 & 6 & -t \end{bmatrix} - 6 \det \begin{bmatrix} 6 & 5 & 0 \\ 0 & -t & 6 \\ 0 & 6 & -t \end{bmatrix}$$
$$\det \begin{bmatrix} -t & 5 & 0 \\ 5 & -t & 6 \\ 0 & 6 & -t \end{bmatrix} = -t(t^2 - 36) - 5(-5t) = -t^3 + 61t$$
$$\det \begin{bmatrix} 6 & 5 & 0 \\ 0 & -t & 6 \\ 0 & 6 & -t \end{bmatrix} = 6(t^2 - 36) = 6t^2 - 216$$
$$\chi_A(t) = -t(-t^3 + 61t) - 6(6t^2 - 216) = t^4 - 97t^2 + 1296.$$

Note that $\chi_A(t)$ can be regarded as a quadratic function of t^2. If $\chi_A(t) = 0$, the quadratic formula tellus us that

$$t^2 = \frac{97 \pm \sqrt{97^2 - 4 \times 1296}}{2} = \frac{97 \pm 65}{2} = 16 \text{ or } 81,$$

so $t = \pm 4$ or $t = \pm 9$. For this kind of question it is standard to list all the positive eigenvalues first, followed by all the negative ones. We thus take

$$\lambda_1 = 4 \qquad \lambda_2 = 9 \qquad \lambda_3 = -4 \qquad \lambda_4 = -9.$$

As there are two strictly positive eigenvalues and two strictly negative ones, we already see that the rank is $2 + 2 = 4$ and the signature is $2 - 2 = 0$. To complete the diagonalisation we need to find eigenvectors corresponding to the above eigenvalues. The relevant row-reductions are as follows:

$$A - 4I = \begin{bmatrix} -4 & 6 & 0 & 0 \\ 6 & -4 & 5 & 0 \\ 0 & 5 & -4 & 6 \\ 0 & 0 & 6 & -4 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 2/3 \\ 0 & 0 & 1 & -2/3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$A - 9I = \begin{bmatrix} -9 & 6 & 0 & 0 \\ 6 & -9 & 5 & 0 \\ 0 & 5 & -9 & 6 \\ 0 & 0 & 6 & -9 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & -3/2 \\ 0 & 0 & 1 & -3/2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$A + 4I = \begin{bmatrix} 4 & 6 & 0 & 0 \\ 6 & 4 & 5 & 0 \\ 0 & 5 & 4 & 6 \\ 0 & 0 & 6 & 4 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 2/3 \\ 0 & 0 & 1 & 2/3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$A + 9I = \begin{bmatrix} 9 & 6 & 0 & 0 \\ 6 & 9 & 5 & 0 \\ 0 & 5 & 9 & 6 \\ 0 & 0 & 6 & 9 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -3/2 \\ 0 & 0 & 1 & 3/2 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

From these we obtain the following eigenvectors:

$$v_1 = \begin{bmatrix} 3 \\ 2 \\ -2 \\ -3 \end{bmatrix} \qquad v_2 = \begin{bmatrix} 2 \\ 3 \\ 3 \\ 2 \end{bmatrix} \qquad v_3 = \begin{bmatrix} 3 \\ -2 \\ -2 \\ 3 \end{bmatrix} \qquad v_4 = \begin{bmatrix} 2 \\ -3 \\ 3 \\ -2 \end{bmatrix}.$$

These all have $v_i.v_i = 26$, so the vectors $u_i = v_i/\sqrt{26}$ form an orthonormal basis. We can now put

$$\begin{aligned}
L_1 &= \sqrt{\lambda_1} u_1 . \begin{bmatrix} w & x & y & z \end{bmatrix}^T = \frac{2}{\sqrt{26}}(3w + 2x - 2y - 3z) \\
L_2 &= \sqrt{\lambda_2} u_2 . \begin{bmatrix} w & x & y & z \end{bmatrix}^T = \frac{3}{\sqrt{26}}(2w + 3x + 3y + 2z) \\
L_3 &= \sqrt{|\lambda_3|} u_3 . \begin{bmatrix} w & x & y & z \end{bmatrix}^T = \frac{2}{\sqrt{26}}(3w - 2x - 2y + 3z) \\
L_4 &= \sqrt{|\lambda_4|} u_4 . \begin{bmatrix} w & x & y & z \end{bmatrix}^T = \frac{3}{\sqrt{26}}(2w - 3x + 3y - 2z),
\end{aligned}$$

and we conclude that

$$Q = L_1^2 + L_2^2 - L_3^2 - L_4^2.$$