Data provided: Formula Sheet

The University Of Sheffield.

SCHOOL OF MATHEMATICS AND STATISTICS   Spring Semester 2006–2007

Mathematics IV (Electrical)

2 Hours

Attempt all FOUR questions.

1 (i) Find the stationary points of the function

$$f(x, y) = x^2y^2 - 4x^2 - 4y^2$$

and determine their nature. ***(15 marks)***

(ii) Use the method of Lagrange multipliers to find the stationary points of the function

$$g(x, y) = y^2 - 8x + 17,$$

subject to the constraint

$$x^2 + y^2 = 9.$$

(10 marks)

2 Evaluate the double integral

$$I_1 = \int_1^\infty \int_{e^{-1}}^1 \frac{1}{x^3 y} \, dy \, dx.$$

(5 marks)

Use the substitution $u = \ln y$ to show that

$$I_2 = \int_{y=0}^{e^{-1}} \frac{1}{y\,(\ln y)^2} = 1.$$

(5 marks)

Hence evaluate the double integral

$$I_3 = \int_{y=0}^{e^{-1}} \int_{x=-\ln y}^{\infty} \frac{1}{x^3 y} \, dy \, dx.$$

(3 marks)

Sketch the region over which the integral I_3 is defined. ***(3 marks)***

Verify that changing the order of integration in the integral I_3 does not change its value.
(9 marks)

3 A vector field $\boldsymbol{A} = \boldsymbol{A}(x, y, z)$ is given by

$$\boldsymbol{A} = \left(4xy - z^3\right) \boldsymbol{i} + 2x^2 \boldsymbol{j} - 3xz^2 \boldsymbol{k}.$$

Calculate $\operatorname{div} \boldsymbol{A}$ and show that $\operatorname{curl} \boldsymbol{A} = 0$. ***(6 marks)***

By evaluating both sides, verify that

$$\nabla^2 \boldsymbol{A} = \operatorname{grad} \operatorname{div} \boldsymbol{A} - \operatorname{curl} \operatorname{curl} \boldsymbol{A}.$$

(11 marks)

Find a scalar field $\phi = \phi(x, y, z)$ such that

$$\boldsymbol{A} = \operatorname{grad} \phi.$$

(8 marks)

4 (i) Use cylindrical co-ordinates to calculate directly the volume integral

$$\iiint_V x^2 \cos z \, dV$$

where V is the volume of the cylinder $0 \leq x^2 + y^2 \leq 4$, $0 \leq z \leq 1$. ***(11 marks)***

(ii) Use cylindrical co-ordinates to evaluate directly the surface integral

$$\iint_S \boldsymbol{G}.\boldsymbol{dS}$$

for the vector field

$$\boldsymbol{G} = x^2 y^2 z \boldsymbol{r}$$

where S is the *curved* surface of the cylinder $0 \leq x^2 + y^2 \leq 4$, $0 \leq z \leq 1$. ***(14 marks)***

End of Question Paper

Formula Sheet for AMA243

Trigonometry

$$\cos(A \pm B) = \cos A \cos B \mp \sin A \sin B$$

$$\sin(A \pm B) = \sin A \cos B \pm \cos A \sin B$$

$$\tan(A \pm B) = \frac{\tan A \pm \tan B}{1 \mp \tan A \tan B}$$

$$a \cos\theta + b \sin\theta = R\cos(\theta - \alpha) \text{ where } R = \sqrt{a^2 + b^2} \text{ and } \cos\alpha = \frac{a}{R},\ \sin\alpha = \frac{b}{R}$$

$$\cos^2\theta = \frac{1}{2}(\cos 2\theta + 1)$$

$$\cos^3\theta = \frac{1}{4}(3\cos\theta + \cos 3\theta)$$

$$\cos^4\theta = \frac{1}{8}(3 + 4\cos 2\theta + \cos 4\theta)$$

$$\sin^2\theta = \frac{1}{2}(1 - \cos 2\theta)$$

$$\sin^3\theta = \frac{1}{4}(3\sin\theta - \sin 3\theta)$$

$$\sin^4\theta = \frac{1}{8}(3 - 4\cos 2\theta + \cos 4\theta)$$