MAS243 — EXAM SOLUTIONS 2006/07

(1) (i) Consider the function $f = x^2y^2 - 4x^2 - 4y^2$. The partial derivatives are $f_x = 2xy^2 - 8x = 2x(y-2)(y+2)$ and $f_y = 2x^2y - 8y = 2(x-2)(x+2)y$, so the critical points are the points where

$$x(y-2)(y+2) = 0 \qquad \text{(A)}$$
$$(x-2)(x+2)y = 0. \qquad \text{(B)}$$

Equation (A) tells us that $x = 0$ or $y = 2$ or $y = -2$. If $x = 0$ then (B) becomes $y = 0$, and if $y = \pm 2$ then (B) becomes $x = \pm 2$. Thus, there are five critical points:

$$P_0 = (0,0) \qquad P_1 = (-2,-2) \qquad P_2 = (-2,+2) \qquad P_3 = (+2,-2) \qquad P_4 = (+2,+2).$$

To find the nature of these critical point, we note that $f_{xx} = 2y^2 - 8$ and $f_{xy} = 4xy$ and $f_{yy} = 2x^2 - 8$, so the Hessian matrix is

$$H = \begin{bmatrix} 2y^2-8 & 4xy \\ 4xy & 2x^2-8 \end{bmatrix}.$$

The top left entry is $A_1 = 2y^2 - 8$ and the determinant is

$$A_2 = (2y^2-8)(2x^2-8) - 16x^2y^2 = 64 - 16x^2 - 16y^2 - 12x^2y^2.$$

At P_0 we have $A_1 = -8 < 0$ and $A_2 = 64 > 0$ so there is a local maximum. At all the other critical points $P_1, \dots, P_4$ we have $A_2 = 64 - 16 \times 4 - 16 \times 4 - 12 \times 4^2 = -256 < 0$ so there is a saddle point.

(ii) We are looking for the critical points of $g(x,y) = y^2 - 8x + 17$ subject to the constraint $h(x,y) = 0$, where $h(x,y) = x^2 + y^2 - 9$. We therefore need to find the unconstrained critical points of the function

$$L(\lambda, x, y) = g(x,y) - \lambda h(x,y) = y^2 - 8x + 17 - \lambda x^2 - \lambda y^2 + 9\lambda.$$

These are the points where the following equations hold:

$$L_\lambda = 9 - x^2 - y^2 \qquad \text{(A)}$$
$$L_x = -8 - 2\lambda x = 0 \qquad \text{(B)}$$
$$L_y = 2y - 2\lambda y = 0. \qquad \text{(C)}$$

Equation (C) gives $(1-\lambda)y = 0$, so either $y = 0$ or $\lambda = 1$. If $\lambda = 1$ then (B) gives $x = -4$, so (A) gives $y^2 = 9 - (-4)^2 = -7$, which is impossible as x and y are supposed to be real. We must therefore have $y = 0$ instead. Substituting this into (A) gives $x = \pm 3$ (and then (B) gives $\lambda = -4/x = \mp 4/3$). Thus, the critical points for the constrained problem are at $(-3,0)$ (where $g = 41$) and $(3,0)$ (where $g = -7$).

(2) (i)

$$\begin{aligned} I_1 &= \int_{x=1}^{\infty} \int_{y=e^{-1}}^{1} \frac{1}{x^3 y}\, dy\, dx = \left(\int_{x=1}^{\infty} x^{-3}\, dx \right) \left(\int_{y=e^{-1}}^{1} y^{-1}\, dy \right) \\ &= \left[\frac{x^{-2}}{-2} \right]_{x=1}^{\infty} \left[\log(y) \right]_{y=e^{-1}}^{1} = (0 - \tfrac{1}{-2})(0 - (-1)) \\ &= 1/2. \end{aligned}$$

(ii) Now consider the integral

$$I_2 = \int_{y=0}^{e^{-1}} \frac{1}{y \log(y)^2}\, dy.$$

We substitute $u = \log(y)$, so $y = e^u$ and $du/dy = y^{-1}$ so $dy = y\,du$. The upper limit $y = e^{-1}$ corresponds to $u = -1$, and as y approaches the lower limit $y = 0$ from above we see that u approaches $-\infty$. This gives

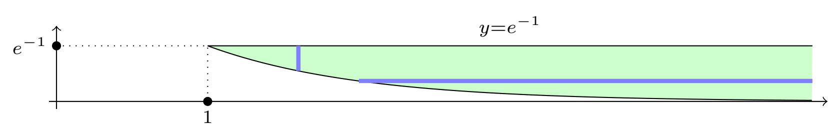
$$I_2 = \int_{u=-\infty}^{-1} \frac{1}{y\,u^2} y\,du = \int_{u=-\infty}^{-1} u^{-2}\,du = \Big[-u^{-1}\Big]_{u=-\infty}^{-1} = -(-1) - 0 = 1.$$

(iii) Now consider $I_3 = \int_{y=0}^{e^{-1}} \int_{x=-\log(y)}^{\infty} \frac{1}{x^3 y} dx\,dy$. The inner integral is

$$\int_{x=-\log(y)}^{\infty} \frac{1}{x^3 y} dx = \left[\frac{-1}{2x^2 y}\right]_{x=-\log(y)}^{\infty} = 0 - \frac{-1}{2(-\log(y))^2 y} = \frac{1}{2y\,\log(y)^2}.$$

This means that the outer integral is just $I_2/2$, so $I_3 = I_2/2 = 1/2$.

(iv) The integral I_3 involves the following region:



The equation of the lower boundary curve can be written as $x = -\log(y)$ or as $y = e^{-x}$. The original formula for I_3 corresponds to dividing the region into horizontal strips, one at each height y from $y = 0$ to $y = e^{-1}$, with the strip at height y running from $x = -\log(y)$ (at the left end) out to $x = +\infty$. We could instead divide the region into vertical strips, one at each horizontal position from $x = 1$ to $x = \infty$. The strip at position x then runs from $y = e^{-x}$ (at the bottom) to $y = e^{-1}$ at the top. This gives

$$I_3 = \int_{x=1}^{\infty} \int_{y=e^{-x}}^{e^{-1}} \frac{1}{x^3 y}\,dy\,dx.$$

The inner integral is

$$\int_{y=e^{-x}}^{e^{-1}} \frac{1}{x^3 y}\,dy = \left[\frac{\log(y)}{x^3}\right]_{y=e_{-x}}^{e^{-1}} = \frac{(-1) - (-x)}{x^3} = x^{-2} - x^{-3}.$$

This gives

$$I_3 = \int_{x=1}^{\infty} x^{-2} - x^{-3}\,dx = \left[\tfrac{1}{2}x^{-2} - x^{-1}\right]_{x=1}^{\infty} = (0-0) - (\tfrac{1}{2} - 1) = 1/2.$$

As expected, this is the same answer as before.

(3) Consider the vector field $\mathbf{A} = (4xy - z^3,\ 2x^2,\ -3xz^2)$.

$$
\begin{aligned}
\mathrm{div}(\mathbf{A}) &= (4xy - z^3)_x + (2x^2)_y + (-3xz^2)_z = 4y - 6xz \\
\mathrm{curl}(\mathbf{A}) &= \det \begin{bmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 4xy - z^3 & 2x^2 & -3xz^2 \end{bmatrix} \\
&= (0 - 0,\ -3z^2 - (-3z^2),\ 4x - 4x) = (0, 0, 0) \\
\nabla^2(4xy - z^3) &= (4xy - z^3)_{xx} + (4xy - z^3)_{yy} + (4xy - z^3)_{zz} = 0 + 0 - 6z = -6z \\
\nabla^2(2x^2) &= (2x^2)_{xx} + (2x^2)_{yy} + (2x^2)_{zz} = 4 + 0 + 0 = 4 \\
\nabla^2(-3xz^2) &= (-3xz^2)_{xx} + (-3xz^2)_{yy} + (-3xz^2)_{zz} = 0 + 0 - 6x = -6x \\
\nabla^2(\mathbf{A}) &= (\nabla^2(4xy - z^3),\ \nabla^2(2x^2),\ \nabla^2(-3xz^2)) = (-6z,\ 4,\ -6x) \\
\mathrm{grad}(\mathrm{div}(\mathbf{A})) &= \mathrm{grad}(4y - 6xz) = ((4y - 6xz)_x,\ (4y - 6xz)_y,\ (4y - 6xz)_z) = (-6z,\ 4,\ -6x) \\
\mathrm{curl}(\mathrm{curl}(\mathbf{A})) &= \mathrm{curl}(0) = 0.
\end{aligned}
$$

From this it is visible that $\nabla^2(\mathbf{A}) = \mathrm{grad}(\mathrm{div}(\mathbf{A})) - \mathrm{curl}(\mathrm{curl}(\mathbf{A}))$.

We now want to find a scalar field ϕ such that $\mathrm{grad}(\phi) = \mathbf{A}$, or equivalently

$$\phi_x = 4xy - z^3 \tag{A}$$

$$\phi_y = 2x^2 \tag{B}$$

$$\phi_z = -3xz^2. \tag{C}$$

Integrating (A) gives

$$\phi = 2x^2y - xz^3 + \alpha, \tag{D}$$

where α depends only on y and z. We now differentiate with respect to y and compare with (B) to get

$$2x^2 + \alpha_y = \phi_y = 2x^2,$$

so $\alpha_y = 0$, so α depends only on z. We now differentiate (D) with respect to z and compare with (C) to get

$$-3xz^2 + \alpha_z = \phi_z = -3xz^2,$$

so $\alpha_z = 0$ as well, so α is completely constant. We can thus take $\alpha = 0$, giving $\phi = 2x^2y - xz^3$.

(4) (i) Let E be the solid cylinder given by $x^2 + y^2 \leq 4$ with $0 \leq z \leq 1$. In cylindrical polar coordinates we have $x = r\cos(\theta)$ and $y = r\sin(\theta)$ so $x^2 + y^2 = r^2$, so the relevant limits are $0 \leq r \leq 2$ and $0 \leq \theta \leq 2\pi$ and $0 \leq z \leq 1$. The volume element is $dV = r\,dr\,d\theta\,dz$. This gives

$$
\begin{aligned}
\iiint_E x^2 \cos(z)\,dV &= \int_{r=0}^{2} \int_{\theta=0}^{2\pi} \int_{z=0}^{1} r^2 \cos^2(\theta) \cos(z) r\,dz\,d\theta\,dr \\
&= \left(\int_{r=0}^{2} r^3\,dr \right) \left(\int_{\theta=0}^{2\pi} \cos^2(\theta)\,d\theta \right) \left(\int_{z=0}^{1} \cos(z)\,dz \right) \\
&= \left[\frac{r^4}{4} \right]_{r=0}^{2} \left[\frac{2\theta + \sin(2\theta)}{4} \right]_{\theta=0}^{2\pi} \Big[\sin(z) \Big]_{z=0}^{1} = \frac{16}{4} \times \pi \times \sin(1) = 4\pi \sin(1).
\end{aligned}
$$

(ii) Now let S be the curved surface of E, and consider the vector field

$$\mathbf{G} = x^2y^2z\mathbf{r} = x^2y^2z(x, y, z) = (x^3y^2z,\ x^2y^3z,\ x^2y^2z^2).$$

We need to evaluate $\iint_S \mathbf{G}.d\mathbf{A} = \iint_S \mathbf{G}.\mathbf{n}\,dA$. Here $\mathbf{n}$ is the outward unit normal to S, which is given by

$$\mathbf{n} = (\cos(\theta),\ \sin(\theta),\ 0).$$

On S we also have $r = 2$ so $x = 2\cos(\theta)$ and $y = 2\sin(\theta)$. Substituting these into the definition of $\mathbf{G}$ gives

$$\begin{aligned}
\mathbf{G} &= (32\cos^3(\theta)\sin^2(\theta)z,\ 32\cos^2(\theta)\sin^3(\theta)z,\ 16\cos^2(\theta)\sin^2(\theta)z^2) \\
\mathbf{G}.\mathbf{n} &= 32\cos^4(\theta)\sin^2(\theta)z + 32\cos^2(\theta)\sin^4(\theta)z \\
&= 32(\cos^2(\theta) + \sin^2(\theta))\cos^2(\theta)\sin^2(\theta)z = 8(2\cos(\theta)\sin(\theta))^2 z \\
&= 8\sin^2(2\theta)z = 4(1 - \cos(4\theta))z.
\end{aligned}$$

It is also standard that $dA = r\,d\theta\,dz = 2\,d\theta\,dz$, so

$$\begin{aligned}
\iint_S \mathbf{G}.\mathbf{n}\,dA &= \int_{z=0}^{1}\int_{\theta=0}^{2\pi} 8(1 - \cos(4\theta))z\,d\theta\,dz \\
&= \left[\frac{z^2}{2}\right]_{z=0}^{1}\left[8\theta - 2\sin(4\theta)\right]_{\theta=0}^{2\pi} = 8\pi.
\end{aligned}$$