The University Of Sheffield.

SCHOOL OF MATHEMATICS AND STATISTICS — Spring Semester 2007–2008

Mathematics IV (Electrical)

2 Hours

Attempt all FOUR questions.

1 (i) The function $F(x, y)$ is defined by

$$F(x, y) = x^2 + kxy + y^2,$$

where k is a constant and $k^2 \neq 4$.

Show that $F(x, y)$ has just one stationary point and find this stationary point.

Determine the nature of the stationary point, considering separately the cases (i) $k^2 > 4$ and (ii) $k^2 < 4$. ***(12 marks)***

(ii) Use the method of Lagrange multipliers to find the maximum and minimum values of the function

$$f(x, y, z) = (x - 2)^2 + (y - 4)^2 + (z - 4)^2$$

subject to the constraint

$$x^2 + y^2 + z^2 = 1.$$

(13 marks)

2 (i) Evaluate the double integral

$$I_1 = \int_0^1 \int_0^x (3 - x - y)\ dy\, dx.$$

(5 marks)

Sketch the region over which the integral I_1 is defined. ***(3 marks)***

(ii) Evaluate I_1 by changing the order of integration. ***(9 marks)***

(iii) Use polar co-ordinates to evaluate the integral

$$I_2 = \iint_R e^{\left(x^2+y^2\right)}\, dx\, dy$$

where R is the semi-circular disk $x^2 + y^2 \leq 4,\ y > 0$. ***(8 marks)***

3 Two scalar fields $U(x, y, z)$ and $V(x, y, z)$ are given by

$$U(x, y, z) = 3x^2 y, \qquad V(x, y, z) = xz^2 - 2y.$$

(i) Find ∇U, ∇V, $\nabla^2 U$ and $\nabla^2 V$. ***(10 marks)***

(ii) Find a unit vector in the direction of the vector $\boldsymbol{n} = (0, 5, 12)$.
Hence find the directional derivatives of U and V in the direction of the vector $\boldsymbol{n}$.
(5 marks)

(iii) A vector field $\boldsymbol{G}$ is defined by

$$\boldsymbol{G} = (\nabla U) \times (\nabla V)\,.$$

Show that $\boldsymbol{G}$ is given by

$$\boldsymbol{G} = 6x^3 z\boldsymbol{i} - 12x^2 yz\boldsymbol{j} + \left(-12xy - 3x^2 z^2\right)\boldsymbol{k}.$$

Find $\operatorname{div}\boldsymbol{G}$ and $\operatorname{curl}\boldsymbol{G}$. ***(10 marks)***

4 A vector field $\boldsymbol{B}$ is given by

$$\boldsymbol{B} = \left(5xy - 6x^2\right)\boldsymbol{i} + (2y - 4x)\,\boldsymbol{j}.$$

Calculate the line integral

$$\int_C \boldsymbol{B.dr}$$

for the following curves C:

(i) The straight line from the origin to the point $(0, 8)$, followed by the straight line from the point $(0, 8)$ to the point $(2, 8)$. ***(11 marks)***

(ii) The straight line from the origin to the point $(2, 8)$. ***(7 marks)***

(iii) The curve $y = x^3$ from the origin to the point $(2, 8)$. ***(7 marks)***

End of Question Paper

Formula Sheet for AMA243 Trigonometry

$$\cos(A \pm B) = \cos A \cos B \mp \sin A \sin B$$

$$\sin(A \pm B) = \sin A \cos B \pm \cos A \sin B$$

$$\tan(A \pm B) = \frac{\tan A \pm \tan B}{1 \mp \tan A \tan B}$$

$$a \cos\theta + b \sin\theta = R\cos(\theta - \alpha) \text{ where } R = \sqrt{a^2 + b^2} \text{ and } \cos\alpha = \frac{a}{R},\ \sin\alpha = \frac{b}{R}$$

$$\cos^2\theta = \frac{1}{2}(\cos 2\theta + 1)$$

$$\cos^3\theta = \frac{1}{4}(3\cos\theta + \cos 3\theta)$$

$$\cos^4\theta = \frac{1}{8}(3 + 4\cos 2\theta + \cos 4\theta)$$

$$\sin^2\theta = \frac{1}{2}(1 - \cos 2\theta)$$

$$\sin^3\theta = \frac{1}{4}(3\sin\theta - \sin 3\theta)$$

$$\sin^4\theta = \frac{1}{8}(3 - 4\cos 2\theta + \cos 4\theta)$$