MAS243 — EXAM SOLUTIONS 2007/08

(1) (i) The partial derivatives are $F_x = 2x + ky$ and $F_y = kx + 2y$, so the critical points are the points where

$$2x + ky = 0 \qquad \text{(A)}$$
$$kx + 2y = 0. \qquad \text{(B)}$$

We can rearrange (B) to give $y = -kx/2$, and substitute this into (A) to get $(2 - k^2/2)x = 0$. By assumption we have $k^2 \neq 4$ so $2 - k^2/2 \neq 0$ so we can divide by this to get $x = 0$. As $y = -kx/2$ we have $y = 0$ as well. This, the only critical point is $(x, y) = (0, 0)$. To find the nature of this critical point, we note that $F_{xx} = 2$ and $F_{xy} = k$ and $F_{yy} = 2$, so the Hessian matrix is $H = \left[\begin{smallmatrix} 2 & k \\ k & 2 \end{smallmatrix}\right]$, so the top left entry is $A_1 = 2$ and the determinant is $A_2 = 4 - k^2$. If $k^2 > 4$ we have $A_2 < 0$ so the critical point is a saddle. If $k^2 < 4$ then $A_1, A_2 > 0$ so we have a local minimum.

(ii) We are looking for the critical points of $f(x, y, z) = (x - 2)^2 + (y - 4)^2 + (z - 4)^2$ subject to the constraint $g(x, y, z) = 0$, where $g(x, y, z) = x^2 + y^2 + z^2 - 1$. We therefore need to find the unconstrained critical points of the function $L(\lambda, x, y, z) = f(x, y, z) - \lambda g(x, y, z)$. These are the points where the following equations hold:

$$L_\lambda = 1 - x^2 - y^2 - z^2 = 0 \qquad \text{(A)}$$
$$L_x = 2x - 4 - 2\lambda x = 0 \qquad \text{(B)}$$
$$L_y = 2y - 8 - 2\lambda y = 0 \qquad \text{(C)}$$
$$L_z = 2z - 8 - 2\lambda z = 0. \qquad \text{(D)}$$

The last three equations can be rearranged as

$$(1 - \lambda)x = 2 \qquad \text{(E)}$$
$$(1 - \lambda)y = 4 \qquad \text{(F)}$$
$$(1 - \lambda)z = 4. \qquad \text{(G)}$$

Dividing (F) by (E) gives $y/x = 2$, so $y = 2x$. Similarly, dividing (G) by (E) gives $z = 2x$. Substituting these into (A) gives

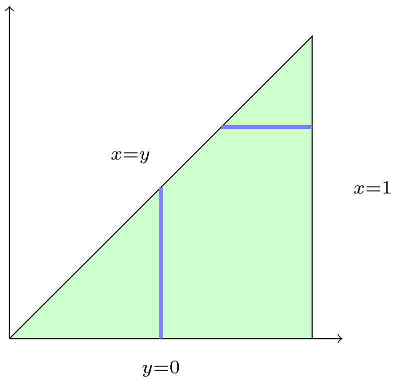
$$1 = x^2 + y^2 + z^2 = x^2 + 4x^2 + 4x^2 = 9x^2,$$

so $x^2 = 1/9$, so $x = \pm 1/3$. We also have $y = z = 2x$, so the critical points are $(1/3, 2/3, 2/3)$ and $(-1/3, -2/3, -2/3)$. The values of f at these points are

$$f(1/3, 2/3, 2/3) = (-5/3)^2 + (-10/3)^2 + (-10/3)^2 = 225/9 = 25$$
$$f(-1/3, -2/3, -2/3) = (-7/3)^2 + (-14/3)^2 + (-14/3)^2 = 441/9 = 49.$$

It follows that the maximum value of f (subject to $g = 0$) is $f = 49$, and the minimum is $f = 25$.

(2) (i) The integral $I = \int_{x=0}^{1} \int_{y=0}^{x} 3 - x - y \, dy \, dx$ involves the following region:



The original formula corresponds to dividing the region into vertical strips, one at each horizontal position from $x=0$ to $x=1$, with the strip at position x running from $y=0$ at the bottom to $y=x$ at the top. We can evaluate the integral as follows:

$$I = \int_{x=0}^{1} \left[3y - xy - \tfrac{1}{2}y^2\right]_{y=0}^{x} dx = \int_{x=0}^{1} 3x - x^2 - \tfrac{1}{2}x^2\, dx = 3\int_{x=0}^{1} x - \tfrac{1}{2}x^2\, dx$$
$$= 3\left[\tfrac{1}{2}x^2 - \tfrac{1}{6}x^3\right]_{x=0}^{1} = 3 \times (\tfrac{1}{2} - \tfrac{1}{6}) = 1.$$

(ii) We could instead divide the region into horizontal strips, one at each height from $y=0$ to $y=1$, with the strip at height y starting from $x=y$ at the left end, and ending at $x=1$ at the right end. This gives

$$I_1 = \int_{y=0}^{1}\int_{x=y}^{1} 3 - x - y\, dx\, dy.$$

Here the inner integral is

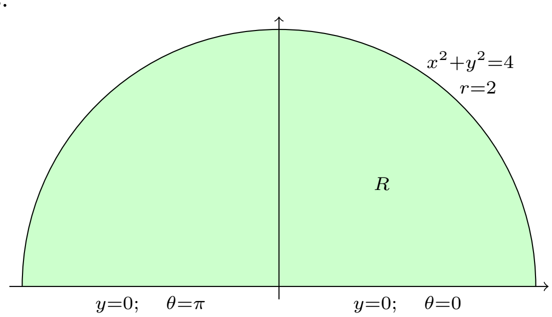
$$\int_{x=y}^{1} 3 - x - y\, dx = \left[3x - \tfrac{1}{2}x^2 - xy\right]_{x=y}^{1} = (3 - \tfrac{1}{2} - y) - (3y - \tfrac{1}{2}y^2 - y^2)$$
$$= \tfrac{3}{2}y^2 - 4y + \tfrac{5}{2}.$$

Putting this into the outer integral gives

$$I_1 = \int_{y=0}^{1} \tfrac{3}{2}y^2 - 4y + \tfrac{5}{2}\, dy = \left[\tfrac{1}{2}y^3 - 2y^2 + \tfrac{5}{2}y\right]_{y=0}^{1}$$
$$= \tfrac{1}{2} - 2 + \tfrac{5}{2} = 1.$$

As expected, this is the same answer as before.

(iii) The region is as follows:



The limits in polar coordinates are $0 \leq r \leq 2$ and $0 \leq \theta \leq \pi$. We now consider the integral

$$I_2 = \iint_R e^{x^2+y^2}\, dx\, dy$$

When we rewrite this in polar coordinates, x^2+y^2 becomes r^2, and the area element $dx\,dy = dA$ becomes $r\,dr\,d\theta$. We thus have

$$I_2 = \int_{\theta=0}^{\pi}\int_{r=0}^{2} e^{r^2}\,r\,dr\,d\theta = \pi\int_{r=0}^{2} e^{r^2}\,r\,dr.$$

If we substitute $u = r^2$, then $du = 2r\,dr$, so $r\,dr = \frac{1}{2}\,du$. Moreover, the limits $r = 0$ and $r = 2$ become $u = 0$ and $u = 4$. We thus have

$$I_2 = \pi\int_{u=0}^{4} e^u \times \tfrac{1}{2}du = \tfrac{1}{2}\pi\left[e^u\right]_{u=0}^{4} = \tfrac{1}{2}\pi(e^4-1).$$

(3) Consider the scalar fields $U = 3x^2y$ and $V = xz^2 - 2y$.

(i)

$$\begin{aligned}
\nabla(U) &= (U_x,\ U_y,\ U_z) = (6xy,\ 3x^2,\ 0)\\
\nabla(V) &= (V_x,\ V_y,\ V_z) = (z^2,\ -2,\ 2xz)\\
\nabla^2(U) &= U_{xx}+U_{yy}+U_{zz} = 6y+0+0 = 6y\\
\nabla^2(V) &= V_{xx}+V_{yy}+V_{zz} = 0+0+2x = 2x.
\end{aligned}$$

(ii) If $\mathbf{n} = (0,\ 5,\ 12)$ then

$$\|\mathbf{n}\| = \sqrt{0^2+5^2+12^2} = \sqrt{0+25+144} = \sqrt{169} = 13,$$

so the unit vector in the direction of $\mathbf{n}$ is

$$\widehat{\mathbf{n}} = \mathbf{n}/\|\mathbf{n}\| = (0,\ 5/13,\ 12/13).$$

The directional deriviatives of U and V in the direction of $\mathbf{n}$ are

$$\begin{aligned}
\widehat{\mathbf{n}}.\nabla(U) &= (0,\ 5/13,\ 12/13).(6xy,\ 3x^2,\ 0) = 15x^2/13\\
\widehat{\mathbf{n}}.\nabla(V) &= (0,\ 5/13,\ 12/13).(z^2, -2, 2xz) = (24xz-10)/13.
\end{aligned}$$

(iii) We now put $\mathbf{G} = (\nabla U)\times(\nabla V)$. This is given by

$$\mathbf{G} = \det\begin{bmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ 6xy & 3x^2 & 0\\ z^2 & -2 & 2xz\end{bmatrix} = (6x^3z,\ -12x^2yz,\ -12xy-3x^2z^2)$$

$$\begin{aligned}
\operatorname{div}(\mathbf{G}) &= \frac{\partial}{\partial x}(6x^3z) + \frac{\partial}{\partial y}(-12x^2yz) + \frac{\partial}{\partial z}(-12xy-3x^2z^2)\\
&= 18x^2z - 12x^2z - 6x^2z = 0\\
\operatorname{curl}(\mathbf{G}) &= \det\begin{bmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z}\\ 6x^3z & -12x^2yz & -12xy-3x^2z^2\end{bmatrix}\\
&= ((-12x)-(-12x^2y),\ (6x^3)-(-12y-6xz^2),\ (-24xyz)-0)\\
&= (12x^2y-12x,\ 12y+6xz^2+6x^3,\ -24xyz).
\end{aligned}$$

(4) Let $\mathbf{B}$ be the two-dimensional vector field $(5xy-6x^2, 2y-4x)$.

(i) Let C_1' be the straight line from $(0,0)$ to $(0,8)$, let C_1'' be the straight line from $(0,8)$ to $(2,8)$, and let C_1 consist of C_1' followed by C_1''. On C_1' we have $x = 0$ so $d\mathbf{r} = (0, dy)$ and $\mathbf{B} = (0, 2y)$, so $\mathbf{B}.d\mathbf{r} = 2y\,dy$. This means that

$$\int_{C_1'}\mathbf{B}.d\mathbf{r} = \int_{y=0}^{8} 2y\,dy = \left[y^2\right]_{y=0}^{8} = 64.$$

Similarly, on C_1'' we have $y = 8$ so $d\mathbf{r} = (dx, 0)$ and $\mathbf{B} = (40x-6x^2,\ 16-4x)$, so $\mathbf{B}.d\mathbf{r} = (40x-6x^2)dx$. This gives

$$\int_{C_1''}\mathbf{B}.d\mathbf{r} = \int_{x=0}^{2} 40x-6x^2dx = \left[20x^2-2x^3\right]_0^2 = 64,$$

so

$$\int_{C_1} \mathbf{B}.d\mathbf{r} = \int_{C_1'} \mathbf{B}.d\mathbf{r} + \int_{C_1''} \mathbf{B}.d\mathbf{r} = 64 + 64 = 128.$$

(ii) Now let C_2 be the straight line from $(0,0)$ to $(2,8)$. This can be parametrised as $\mathbf{r} = (x,y) = (2t, 8t)$ for $0 \leq t \leq 1$. Thus, on C_2 we have

$$\begin{aligned}
d\mathbf{r} &= (2,8)\,dt \\
\mathbf{B} &= (5 \times (2t) \times (8t) - 6 \times (2t)^2, \quad 2 \times 8t - 4 \times 2t) \\
&= (56t^2,\ 8t) \\
\mathbf{B}.d\mathbf{r} &= (56t^2,\ 8t).(2,8)\,dt = (112t^2 + 64t)\,dt \\
\int_{C_2} \mathbf{B}.d\mathbf{r} &= \int_{t=0}^{1} (112t^2 + 64t)\,dt = \left[\tfrac{112}{3}t^3 + 32t^2\right]_{t=0}^{1} \\
&= \tfrac{112}{3} + 32 = \tfrac{208}{3}.
\end{aligned}$$

(iii) Finally, let C_3 be the curve given by $y = x^3$ for $0 \leq x \leq 2$. This has

$$\begin{aligned}
d\mathbf{r} &= (dx,\ dy) = (1,\ 3x^2)dx \\
\mathbf{B} &= (5 \times x \times x^3 - 6x^2, \quad 2x^3 - 4x) = (5x^4 - 6x^2, \quad 2x^3 - 4x) \\
\mathbf{B}.d\mathbf{r} &= ((5x^4 - 6x^2) + 3x^2(2x^3 - 4x))dx = (6x^5 + 5x^4 - 12x^3 - 6x^2)dx \\
\int_{C_3} \mathbf{B}.d\mathbf{r} &= \int_{x=0}^{2} (6x^5 + 5x^4 - 12x^3 - 6x^2)dx = \left[x^6 + x^5 - 3x^4 - 2x^3\right]_{x=0}^{2} \\
&= 64 + 32 - 48 - 16 = 32.
\end{aligned}$$