MAS243 — EXAM SOLUTIONS 2008/09

(1) (i) The partial derivatives are $F_x = 2x + 4xy$ and $F_y = 2y + 2x^2 - 3$, so the critical points are the points where

$$2x + 4xy = 0 \tag{A}$$
$$2y + 2x^2 - 3 = 0. \tag{B}$$

We can rearrange (B) to give $y = \frac{3}{2} - x^2$, and substitute this into (A) to get $2x + 6x - 4x^3 = 0$, and this factors as $4x(2 - x^2) = 0$, so the solutions for x are $x = 0$, $x = \sqrt{2}$ and $x = -\sqrt{2}$. Using $y = \frac{3}{2} - x^2$ we see that $y = \frac{3}{2}$ when $x = 0$, and $y = -\frac{1}{2}$ when $x = \pm\sqrt{2}$. Thus, the three critical points are

$$P = (0, \tfrac{3}{2}) \qquad Q = (\sqrt{2}, -\tfrac{1}{2}) \qquad R = (-\sqrt{2}, -\tfrac{1}{2}).$$

To classify these, we need to use the Hessian

$$H = \begin{bmatrix} F_{xx} & F_{xy} \\ F_{yx} & F_{yy} \end{bmatrix} = \begin{bmatrix} 2+4y & 4x \\ 4x & 2 \end{bmatrix}.$$

The top left entry is $A_1 = 2+4y$, and the determinant is $A_2 = 2(2+4y) - (4x)^2 = 4 + 8y - 16x^2$. At P we have $A_1 = 8 > 0$ and $A_2 = 16 > 0$ so there is a local minimum. At Q and R we have $A_2 = -32 < 0$ so there is a saddle point.

(ii) We are looking for the critical points of $f(x, y, z) = (x + 1)^2 + (2y + 1)^2 + (3z + 1)^2$ subject to the constraint $g(x, y, z) = 0$, where $g(x, y, z) = x^2 + 4y^2 + 9z^2 - 12$. We therefore need to find the unconstrained critical points of the function $L(\lambda, x, y, z) = f(x, y, z) - \lambda g(x, y, z)$. These are the points where the following equations hold:

$$L_\lambda = 12 - x^2 - 4y^2 - 9z^2 = 0 \tag{A}$$
$$L_x = 2x + 2 - 2\lambda x = 0 \tag{B}$$
$$L_y = 8y + 4 - 8\lambda y = 0 \tag{C}$$
$$L_z = 18z + 6 - 18\lambda z = 0. \tag{D}$$

The last three equations can be rearranged as

$$(\lambda - 1)x = 1 \tag{E}$$
$$(\lambda - 1)y = \tfrac{1}{2} \tag{F}$$
$$(\lambda - 1)z = \tfrac{1}{3}. \tag{G}$$

Dividing (E) by (F) gives $y = x/2$. Similarly, dividing (E) by (G) gives $z = x/3$. Substituing these into (A) gives

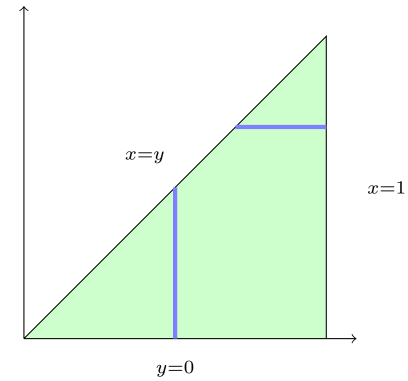
$$12 = x^2 + 4y^2 + 9z^2 = x^2 + x^2 + x^2 = 3x^2,$$

so $x^2 = 4$, so $x = \pm 2$. If $x = 2$ then $y = x/2 = 1$ and $z = x/3 = 2/3$, and similarly when $x = -1$. Thus, the constrained critical points are $(x, y, z) = (2, 1, 2/3)$ and $(x, y, z) = (-2, -1, -2/3)$. The values of f at these points are

$$f(2, 1, \tfrac{2}{3}) = 3^2 + 3^2 + 3^2 = 27$$
$$f(-2, -1, -\tfrac{2}{3}) = (-1)^2 + (-1)^2 + (-1)^2 = 3.$$

It follows that the maximum value of f (subject to $g = 0$) is $f = 27$, and the minimum is $f = 3$.

(2) (i) The integral $I = \int_{y=0}^{1} \int_{x=y}^{1} 4x^6 e^{x^3 y}\, dx\, dy$ involves the following region:



The original formula corresponds to dividing the region into horizontal strips, one at each height from $y = 0$ to $y = 1$, with the strip at height y starting from $x = y$ at the left end, and ending at $x = 1$ at the right end. We can instead divide the region into vertical strips, one at each horizontal position from $x = 0$ to $x = 1$, with the strip at position x running from $y = 0$ at the bottom to $y = x$ at the top. This gives

$$I = \int_{x=0}^{1} \int_{y=0}^{x} 4x^6 e^{x^3 y}\, dy\, dx.$$

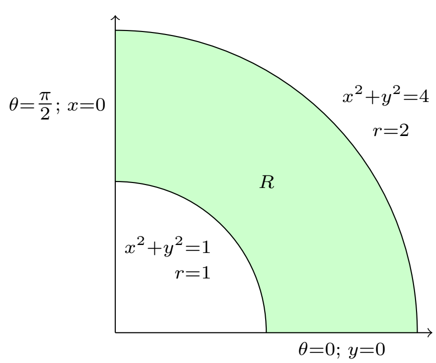
For the inner integral, we note that $\int e^{ay}\, dy = a^{-1}e^{ay}$ whenever a is independent of y. Taking $a = x^3$, we get

$$\begin{aligned}\int_{y=0}^{x} 4x^6 e^{x^3 y}\, dy &= \left[4x^6 x^{-3} e^{x^3 y}\right]_{y=0}^{x} = \left[4x^3 e^{x^3 y}\right]_{y=0}^{x} \\ &= 4x^3 e^{x^4} - 4x^3 e^0 = 4x^3 e^{x^4} - 4x^3.\end{aligned}$$

Note that the derivative of x^4 is $4x^3$, so (by the chain rule) we have $\frac{d}{dx}(e^{x^4}) = 4x^3 e^{x^4}$, which means that $\int 4x^3 e^{x^4}\, dx = e^{x^4}$. (This approach is essentially the same as doing the integral by substituting $u = x^4$.) Feeding this into the outer integral, we get

$$\begin{aligned}I &= \int_{x=0}^{1} 4x^3 e^{x^4} - 4x^3\, dy = \left[e^{x^4} - x^4\right]_{x=0}^{1} \\ &= (e^1 - 1) - (e^0 - 0) = e - 2.\end{aligned}$$

(ii) The region is as follows:



The limits in polar coordinates are $1 \leq r \leq 2$ and $0 \leq \theta \leq \frac{\pi}{2}$. We now consider the integral

$$I = \iint_R (1 + x)\sqrt{x^2 + y^2}\, dA$$

When we rewrite this in polar coordinates, $\sqrt{x^2+y^2}$ becomes r, and dA becomes $r\,dr\,d\theta$, and x becomes $r\cos(\theta)$. We thus have

$$I = \int_{r=1}^{2}\int_{\theta=0}^{\frac{\pi}{2}} (1+r\cos(\theta))r\,r\,d\theta\,dr$$
$$= \int_{r=1}^{2}\int_{\theta=0}^{\frac{\pi}{2}} r^2\,d\theta\,dr + \int_{r=1}^{2}\int_{\theta=0}^{\frac{\pi}{2}} r^3\cos(\theta)\,d\theta\,dr$$
$$\int_{r=1}^{2}\int_{\theta=0}^{\frac{\pi}{2}} r^2\,d\theta\,dr = \frac{\pi}{2}\left[\tfrac{1}{3}r^3\right]_1^2 = \frac{\pi}{6}(8-1) = \frac{7\pi}{6}$$
$$\int_{r=1}^{2}\int_{\theta=0}^{\frac{\pi}{2}} r^3\cos(\theta)\,d\theta\,dr = \left[\frac{r^4}{4}\right]_{r=1}^{2}\left[\sin(\theta)\right]_{\theta=0}^{\frac{\pi}{2}} = \frac{15}{4}(1-0) = \frac{15}{4}$$
$$I = \frac{7\pi}{6} + \frac{15}{4}.$$

(3)

$$\mathbf{A} = (8xz - z^2,\quad 3y^2,\quad ax^2 - 2xz)$$
$$\mathrm{div}(\mathbf{A}) = \frac{\partial}{\partial x}(8xz - z^2) + \frac{\partial}{\partial y}(3y^2) + \frac{\partial}{\partial z}(8x^2 - 2xz) = 8z + 6y - 2x$$
$$\mathrm{curl}(\mathbf{A}) = \det\begin{bmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 8xz - z^2 & 3y^2 & ax^2 - 2xz \end{bmatrix}$$
$$= (0-0, (8x-2z)-(2ax-2z), 0-0) = (0, (8-2a)x, 0)$$
$$\mathrm{grad}\,\mathrm{div}(\mathbf{A}) = \mathrm{grad}(8z+6y-2x) = \left(\frac{\partial}{\partial x}(8z+6y-2x), \frac{\partial}{\partial y}(8z+6y-2x), \frac{\partial}{\partial z}(8z+6y-2x)\right)$$
$$= (-2,\ 6,\ 8)$$
$$\mathrm{curl}\,\mathrm{curl}(\mathbf{A}) = \det\begin{bmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 0 & (8-2a)x & 0 \end{bmatrix} = (0-0,\ 0-0,\ (8-2a)-0)$$
$$= (0, 0, 8-2a)$$
$$\nabla^2\mathbf{A} = (\nabla^2(8xz - z^2),\ \nabla^2(3y^2),\ \nabla^2(ax^2 - 2xz)) = (-2,\ 6,\ 2a)$$
$$= \mathrm{grad}\,\mathrm{div}(\mathbf{A}) - \mathrm{curl}\,\mathrm{curl}(\mathbf{A}).$$

Next, from the formula $\mathrm{curl}(\mathbf{A}) = (0,\ (8-2a)x,\ 0)$ it is clear that $a = 4$ is the only value for which $\mathrm{curl}(\mathbf{A}) = 0$. In this case we have $\mathbf{A} = (8xz - z^2,\quad 3y^2,\quad 4x^2 - 2xz)$. We want to find a potential function $\phi(x, y, z)$ with $\mathrm{grad}(\phi) = \mathbf{A}$, or in other words

$$\phi_x = 8xz - z^2 \qquad \text{(A)}$$
$$\phi_y = 3y^2 \qquad \text{(B)}$$
$$\phi_z = 4x^2 - 2xz \qquad \text{(C)}$$

Integrating (A) gives $\phi = 4x^2z - xz^2 + \psi$, where ψ is constant with respect to x, or in other words depends only on y and z. From this we get $\phi_y = (4x^2z - xz^2 + \psi)_y = \psi_y$, so (B) gives $\psi = y^3 + \chi$. Here χ is constant with respect to both x and y, so it depends only on z. Substituting $\psi = y^3 + \chi$ into $\phi = 4x^2z - xz^2 + \psi$ gives $\phi = 4x^2z - xz^2 + y^3 + \chi$, so $\phi_z = 4x^2 - 2xz + \chi_z$. However, equation (C) tells us that $\phi_z = 4x^2 - 2xz$, so $\chi_z = 0$. As χ depends only on z and $\chi_z = 0$, we see that χ is an absolute constant, and we can take it to be zero. This gives $\phi = 4x^2z - xz^2 + y^3$ as a potential function.

(4) (i) The top surface S_1 is given in spherical polar coordinates by

$$(x, y, z) = (\sin(\phi)\cos(\theta),\ \sin(\phi)\sin(\theta),\ \cos(\phi))$$

with $0 \leq \phi \leq \pi/2$ and $0 \leq \theta \leq 2\pi$. The outward unit normal $\mathbf{n}$ points directly away from the origin and has length one so it is just the same as (x, y, z). The vector field is

$$\mathbf{E} = (x, y, 1 - z) = (\sin(\phi)\cos(\theta),\ \sin(\phi)\sin(\theta),\ 1 - \cos(\phi)),$$

so

$$\begin{aligned}\mathbf{E}.\mathbf{n} &= \sin^2(\phi)\cos^2(\theta) + \sin^2(\phi)\sin^2(\theta) + \cos(\phi) - \cos^2(\phi) \\ &= \sin^2(\phi) + \cos(\phi) - \cos^2(\phi) = 2\sin^2(\phi) + \cos(\phi) - 1\end{aligned}$$

The area element is $dA = \sin(\phi)\,d\phi\,d\theta$, so

$$\begin{aligned}\mathbf{E}.\mathbf{n}dA &= (2\sin^3(\phi) + \sin(\phi)\cos(\phi) - \sin(\phi))d\phi\,d\theta \\ &= \left(\tfrac{3}{2}\sin(\phi) - \tfrac{1}{2}\sin(3\phi) + \tfrac{1}{2}\sin(2\phi) - \sin(\phi)\right)d\phi\,d\theta \\ &= \frac{1}{2}\left(\sin(\phi) - \sin(3\phi) + \sin(2\phi)\right)d\phi\,d\theta \\ \iint_{S_1}\mathbf{E}.\mathbf{n}\,dA &= \int_{\phi=0}^{\pi/2}\int_{\theta=0}^{2\pi}\frac{1}{2}\left(\sin(\phi) - \sin(3\phi) + \sin(2\phi)\right)d\theta\,d\phi \\ &= \pi\int_{\phi=0}^{\pi/2}\left(\sin(\phi) - \sin(3\phi) + \sin(2\phi)\right)d\phi \\ &= \pi\left[-\cos(\phi) + \tfrac{1}{3}\cos(3\phi) - \tfrac{1}{2}\cos(2\phi)\right]_{\phi=0}^{\pi/2} \\ &= \pi((-0 + \tfrac{1}{3}(0) - \tfrac{1}{2}(-1)) - (-1 + \tfrac{1}{3} - \tfrac{1}{2})) \\ &= \pi(\tfrac{1}{2} - \tfrac{-7}{6}) = \frac{5\pi}{3}\end{aligned}$$

On the bottom surface S_2, we have $z = 0$ and the outward unit normal is $\mathbf{n} = (0, 0, -1)$, so

$$\mathbf{E}.\mathbf{n} = (x, y, 1 - z).(0, 0, -1) = z - 1 = -1.$$

It follows that

$$\iint_{S_2}\mathbf{E}.\mathbf{n}\,dA = -\iint_{S_2}dA = -\text{area}(S_2) = -\pi.$$

This gives

$$\iint_S\mathbf{E}.\mathbf{n}\,dA = \iint_{S_1}\mathbf{E}.\mathbf{n}\,dA + \iint_{S_2}\mathbf{E}.\mathbf{n}\,dA = \frac{5\pi}{3} - \pi = \frac{2\pi}{3}.$$

(ii) Next, we have

$$\text{div}(\mathbf{E}) = \frac{\partial}{\partial x}(x) + \frac{\partial}{\partial y}(y) + \frac{\partial}{\partial z}(1 - z) = 1 + 1 + (-1) = 1$$

so $\iiint_V \text{div}(\mathbf{E})\,dV = \text{volume}(V)$. The volume of a sphere of radius a is $\frac{4}{3}\pi a^3$. Here we have half of a sphere of radius one, so the volume is $\frac{2\pi}{3}$. More explicitly, the volume element is $dV = r^2\sin(\phi)\,dr\,d\phi\,d\theta$, and the relevant limits are $0 \leq \phi \leq \pi/2$ and $0 \leq \theta \leq 2\pi$ (as before) and $0 \leq r \leq 1$. This gives the volume as

$$\begin{aligned}\iiint_V dV &= \int_{\phi=0}^{\pi/2}\int_{\theta=0}^{2\pi}\int_{r=0}^{1} r^2\sin(\phi)\,dr\,d\theta\,d\phi \\ &= 2\pi\left(\int_{\phi=0}^{\pi/2}\sin(\phi)\,d\phi\right)\left(\int_{r=0}^{1} r^2\,dr\right) \\ &= 2\pi\left[-\cos(\phi)\right]_{\phi=0}^{\pi/2}\left[\tfrac{1}{3}r^3\right]_{r=0}^{1} = \frac{2\pi}{3}(0 - (-1)) = \frac{2\pi}{3}.\end{aligned}$$

This is the same as the answer to part (i), as expected.