Data provided: Formula Sheet

The University Of Sheffield.

SCHOOL OF MATHEMATICS AND STATISTICS **Spring Semester 2009–2010**

Mathematics IV (Electrical)

2 **Hours**

Attempt all FOUR questions.

1 (i) Find and classify the stationary points of the function

$$F(x, y) = \frac{1}{y} - \frac{1}{x} - 4x + 9y.$$

(12 marks)

(ii) Use the method of Lagrange multipliers to find the maximum and minimum values attained by the function

$$f(x, y, z) = 3x + 4y + 5z$$

subject to the constraint

$$x^2 + y^2 + z^2 = 50.$$

(13 marks)

2 (i) Sketch the region over which the integral I_1 is defined, where

$$I_1 = \iint\limits_R \left(3y^2 + 10x^4 y\right)\, dx\, dy,$$

and R is the rectangle with vertices $(0,0), (1,0), (1,2)$ and $(0,2)$.
Evaluate I_1. ***(6 marks)***

(ii) Sketch the region over which the integral I_2 is defined, where

$$I_2 = \int_0^{\sqrt{\pi}} \int_{x^2}^{\pi} x \cos\left(x^2\right)\, dy\, dx.$$

Evaluate I_2 by changing the order of integration. ***(10 marks)***

(iii) Sketch the region over which the integral I_3 is defined, where

$$I_3 = \iint\limits_D \frac{1}{1 + x^2 + y^2}\, dx\, dy,$$

and D is the region consisting of all points (x, y) such that $x \geq 0$, $y \geq 0$ and $x^2 + y^2 \leq 4$.
Evaluate I_3. ***(9 marks)***

3 (i) A scalar field ϕ is given by

$$\phi(x, y, z) = 2xz^4 - x^2 y.$$

Find $\operatorname{grad}\phi$ and $\nabla^2\phi$ at the point $(2, -2, 1)$. ***(8 marks)***

(ii) A vector field $\boldsymbol{A}$ is given by

$$\boldsymbol{A} = 2x^2\boldsymbol{i} - 3yz\boldsymbol{j} + xz^2\boldsymbol{k}.$$

Find $\operatorname{div}\boldsymbol{A}$ and $\operatorname{curl}\boldsymbol{A}$. ***(7 marks)***

By evaluating both sides, verify that

$$\nabla^2\boldsymbol{A} = \operatorname{grad}\operatorname{div}\boldsymbol{A} - \operatorname{curl}\operatorname{curl}\boldsymbol{A}.$$

(10 marks)

4 Let $\boldsymbol{F}$ be the vector field

$$\boldsymbol{F} = 3y\boldsymbol{i} + 4x\boldsymbol{j} + 2z^2\boldsymbol{k}.$$

Let C be the closed curve given by $x^2+y^2=9$ and $z=0$ and let S_1 be the surface of the open hemisphere $x^2+y^2+z^2=9$, $z \geq 0$. Let S_2 be the disc given by $x^2+y^2 \leq 9$ and $z=0$ and let S be the surface of the closed hemisphere formed by S_1 and S_2. Let V be the volume enclosed by S.

(i) Let I be the line integral

$$I = \oint_C \boldsymbol{F.dr}$$

and let J be the surface integral

$$J = \iint_{S_1} \operatorname{curl} \boldsymbol{F.dS}.$$

Evaluate I and J separately and verify that, in accordance with Stokes' Theorem, $I = J$. ***(16 marks)***

(ii) Let K be the volume integral

$$K = \iiint_V \operatorname{div} \boldsymbol{F}\, dV.$$

Evaluate K and use Gauss' Divergence Theorem to write down the value of the surface integral

$$L = \iint_S \boldsymbol{F.dS}.$$

(9 marks)

End of Question Paper

Formula Sheet for MAS243
Trigonometry

$$\cos(A \pm B) = \cos A \cos B \mp \sin A \sin B$$

$$\sin(A \pm B) = \sin A \cos B \pm \cos A \sin B$$

$$\tan(A \pm B) = \frac{\tan A \pm \tan B}{1 \mp \tan A \tan B}$$

$$a \cos\theta + b \sin\theta = R\cos(\theta - \alpha) \text{ where } R = \sqrt{a^2 + b^2} \text{ and } \cos\alpha = \frac{a}{R},\ \sin\alpha = \frac{b}{R}$$

$$\cos^2\theta = \frac{1}{2}(\cos 2\theta + 1)$$

$$\cos^3\theta = \frac{1}{4}(3\cos\theta + \cos 3\theta)$$

$$\cos^4\theta = \frac{1}{8}(3 + 4\cos 2\theta + \cos 4\theta)$$

$$\sin^2\theta = \frac{1}{2}(1 - \cos 2\theta)$$

$$\sin^3\theta = \frac{1}{4}(3\sin\theta - \sin 3\theta)$$

$$\sin^4\theta = \frac{1}{8}(3 - 4\cos 2\theta + \cos 4\theta)$$