The University Of Sheffield.

SCHOOL OF MATHEMATICS AND STATISTICS **Spring Semester 2009–2010**

Mathematics IV (Electrical)

2 **Hours**

Solutions to examination questions

1 (i) The stationary points of the function $F(x,y)$ occur when

$$\frac{\partial F}{\partial x} = 0 = \frac{\partial F}{\partial y}.$$

In this case we have

$$\frac{\partial F}{\partial x} = \frac{1}{x^2} - 4, \qquad \frac{\partial F}{\partial y} = -\frac{1}{y^2} + 9.$$

Therefore, at a stationary point:

$$x^2 = \frac{1}{4} \text{ and } y^2 = \frac{1}{9}.$$

Therefore the function has four stationary points at $\left(\frac{1}{2}, \frac{1}{3}\right)$, $\left(\frac{1}{2}, -\frac{1}{3}\right)$, $\left(-\frac{1}{2}, \frac{1}{3}\right)$ and $\left(-\frac{1}{2}, -\frac{1}{3}\right)$.

To classify the stationary points, consider the quantity

$$\Delta = \frac{\partial^2 F}{\partial x^2}\frac{\partial^2 F}{\partial y^2} - \left(\frac{\partial^2 F}{\partial x \partial y}\right)^2.$$

In this case, we have

$$\frac{\partial^2 F}{\partial x^2} = -\frac{2}{x^3}, \qquad \frac{\partial^2 F}{\partial x \partial y} = 0, \qquad \frac{\partial^2 F}{\partial y^2} = \frac{2}{y^3}.$$

Hence

$$\Delta = -\frac{4}{x^3 y^3}.$$

At $\left(\frac{1}{2}, \frac{1}{3}\right)$ and $\left(-\frac{1}{2}, -\frac{1}{3}\right)$, we have $\Delta < 0$ so these are saddles.

At $\left(\frac{1}{2}, -\frac{1}{3}\right)$ and $\left(-\frac{1}{2}, \frac{1}{3}\right)$, we have $\Delta > 0$ so each of these points is either a maximum or a minimum.

At $\left(\frac{1}{2}, -\frac{1}{3}\right)$, we have $\frac{\partial^2 F}{\partial x^2} < 0$ so this a maximum.

At $\left(-\frac{1}{2}, \frac{1}{3}\right)$, we have $\frac{\partial^2 F}{\partial x^2} > 0$ so this a minimum. ***(12 marks)***

 Question 1 continued on next page

(ii) The stationary points of the function subject to the constraint are the same as the stationary points of the function

$$L(x, y, z, \lambda) = 3x + 4y + 5z + \lambda\left(x^2 + y^2 + z^2 - 50\right).$$

The stationary points occur when

$$0 = \frac{\partial L}{\partial x} = 3 + 2\lambda x; \quad (1)$$

$$0 = \frac{\partial L}{\partial y} = 4 + 2\lambda y; \quad (2)$$

$$0 = \frac{\partial L}{\partial z} = 5 + 2\lambda z; \quad (3)$$

$$0 = \frac{\partial L}{\partial \lambda} = x^2 + y^2 + z^2 - 50. \quad (4)$$

From equations (1–3) we have that $\lambda \neq 0$ and

$$x = -\frac{3}{2\lambda}, \qquad y = -\frac{4}{2\lambda}, \qquad z = -\frac{5}{2\lambda}.$$

Substituting these values into equation (4) gives

$$50 = \frac{9 + 16 + 25}{4\lambda^2} = \frac{50}{4\lambda^2} \qquad \Rightarrow \qquad 4\lambda^2 = 1$$

and $\lambda = \pm\frac{1}{2}$.

If $\lambda = \frac{1}{2}$, then we have $x = -3$, $y = -4$, $z = -5$, while if $\lambda = -\frac{1}{2}$, then we have $x = 3$, $y = 4$, $z = 5$.

Thus there are two stationary points.

The constraint imposes upper and lower bounds on x, y, z and, as a consequence, on $f(x, y, z)$. So the stationary points must be a maximum and a minimum.

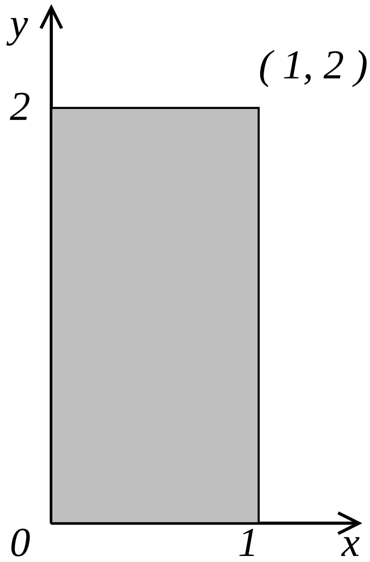
When $x = 3$, $y = 4$ and $z = 5$, we have $f(x, y, z) = 3x + 4y + 5z = 50$.

When $x = -3$, $y = -4$ and $z = -5$, then $f(x, y, z) = 3x + 4y + 5z = -50$.

So the maximum value of $f(x, y, z)$ subject to the given constraint is 50 and the minimum is -50. ***(13 marks)***

2 (i) To evaluate the integral, we first sketch the rectangle.

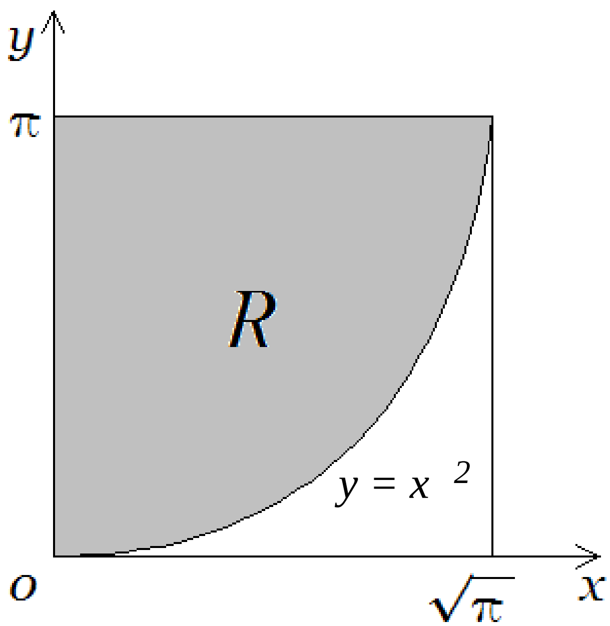
Sketch:



$$
\begin{aligned}
I_1 &= \iint_R \left(3y^2 + 10x^4 y\right)\, dx\, dy \\
&= \int_{y=0}^{2} \int_{x=0}^{1} \left(3y^2 + 10x^4 y\right) dx\, dy \\
&= \int_{y=0}^{2} \left[3xy^2 + 2x^5 y\right]_{x=0}^{1} dy \\
&= \int_{y=0}^{2} \left(3y^2 + 2y\right)\, dy \\
&= \left[y^3 + y^2\right]_{y=0}^{2} \\
&= 8 + 4 = 12.
\end{aligned}
$$

Therefore $I_1 = 12$. ***(6 marks)***

(ii) Sketch of the region of integration R:



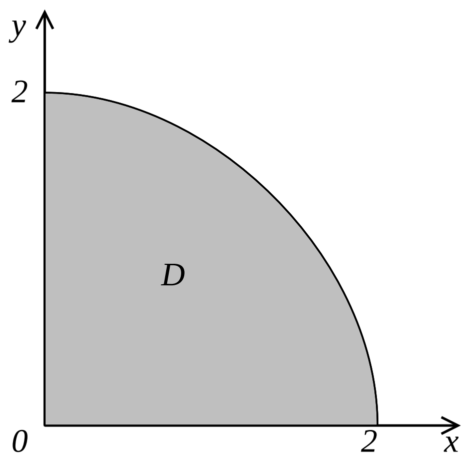
Changing the order of integration,

$$\begin{aligned} I_2 &= \int_{y=0}^{\pi} \int_{x=0}^{\sqrt{y}} x \cos\left(x^2\right)\, dx\, dy \\ &= \int_{y=0}^{\pi} \left[\frac{\sin\left(x^2\right)}{2}\right]_{x=0}^{\sqrt{y}} dy \\ &= \int_{y=0}^{\pi} \frac{\sin y}{2}\, dy \\ &= \left[-\frac{\cos y}{2}\right]_{y=0}^{\pi} \\ &= \frac{1}{2} - \frac{-1}{2} = 1. \end{aligned}$$

(10 marks)

(iii) Using polar co-ordinates, $x = r\cos\theta$ and $y = r\sin\theta$, so that $x^2 + y^2 = r^2$. We also have $dx\, dy = r\, dr\, d\theta$.

Sketch of the region of integration D:



$$\begin{aligned} I_3 &= \int_{\theta=0}^{\frac{\pi}{2}} \int_{r=0}^{2} \frac{1}{1+r^2} r\, dr\, d\theta \\ &= \int_{\theta=0}^{\frac{\pi}{2}} \int_{r=0}^{2} \frac{r}{1+r^2} dr\, d\theta \\ &= \int_{\theta=0}^{\frac{\pi}{2}} \left[\frac{\ln\left(1+r^2\right)}{2}\right]_{r=0}^{2} d\theta \\ &= \int_{\theta=0}^{\frac{\pi}{2}} \left(\frac{\ln 5 - \ln 1}{2}\right) d\theta \\ &= \int_{\theta=0}^{\frac{\pi}{2}} \frac{\ln 5}{2}\, d\theta \\ &= \left[\frac{\ln 5}{2}\theta\right]_{\theta=0}^{\frac{\pi}{2}} \\ &= \frac{\pi \ln 5}{4}. \end{aligned}$$

Therefore

$$I_3 = \frac{\pi \ln 5}{4}.$$

(9 marks)

3 (i) Firstly,

$$\operatorname{grad}\phi = \left(\frac{\partial\phi}{\partial x}, \frac{\partial\phi}{\partial y}, \frac{\partial\phi}{\partial z}\right) = \left(2z^4 - 2xy, -x^2, 8xz^3\right).$$

When $x = 2$, $y = -2$ and $z = 1$, this gives

$$\operatorname{grad}\phi = (10, -4, 16).$$

Secondly,

$$\nabla^2\phi = \frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} + \frac{\partial^2\phi}{\partial z^2} = -2y + 0 + 24xz^2 = 24xz^2 - 2y.$$

At the point $(2, -2, 1)$ we then have

$$\nabla^2\phi = 52.$$

(8 marks)

(ii)

$$\begin{aligned} \operatorname{div}\boldsymbol{A} &= \frac{\partial A_1}{\partial x} + \frac{\partial A_2}{\partial y} + \frac{\partial A_3}{\partial z} \\ &= \frac{\partial}{\partial x}\left(2x^2\right) + \frac{\partial}{\partial y}(-3yz) + \frac{\partial}{\partial z}\left(xz^2\right) = 4x - 3z + 2xz. \end{aligned}$$

$$\begin{aligned} \operatorname{curl}\boldsymbol{A} &= \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ A_1 & A_2 & A_3 \end{vmatrix} = \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ 2x^2 & -3yz & xz^2 \end{vmatrix} \\ &= \left[\frac{\partial}{\partial y}\left(xz^2\right) - \frac{\partial}{\partial z}(-3yz)\right]\boldsymbol{i} + \left[\frac{\partial}{\partial z}\left(2x^2\right) - \frac{\partial}{\partial x}\left(xz^2\right)\right]\boldsymbol{j} \\ &\quad + \left[\frac{\partial}{\partial x}(-3yz) - \frac{\partial}{\partial y}\left(2x^2\right)\right]\boldsymbol{k} \\ &= 3y\boldsymbol{i} - z^2\boldsymbol{j}. \end{aligned}$$

(7 marks)

 Question 3 continued on next page

$$\operatorname{grad}\operatorname{div}\boldsymbol{A} = \left(\frac{\partial}{\partial x}[\operatorname{div}\boldsymbol{A}], \frac{\partial}{\partial y}[\operatorname{div}\boldsymbol{A}], \frac{\partial}{\partial z}[\operatorname{div}\boldsymbol{A}]\right) = (4+2z)\,\boldsymbol{i} + (-3+2x)\,\boldsymbol{k}.$$

$$\begin{aligned}
\operatorname{curl}\operatorname{curl}\boldsymbol{A} &= \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ 3y & -z^2 & 0 \end{vmatrix} \\
&= \left[\frac{\partial}{\partial y}(0) - \frac{\partial}{\partial z}\left(-z^2\right)\right]\boldsymbol{i} + \left[\frac{\partial}{\partial z}(3y) - \frac{\partial}{\partial x}(0)\right]\boldsymbol{j} \\
&\quad + \left[\frac{\partial}{\partial x}\left(-z^2\right) - \frac{\partial}{\partial y}(3y)\right]\boldsymbol{k} \\
&= 2z\boldsymbol{i} - 3\boldsymbol{k}.
\end{aligned}$$

Altogether then,

$$\operatorname{grad}\operatorname{div}\boldsymbol{A} - \operatorname{curl}\operatorname{curl}\boldsymbol{A} = (4+2z)\,\boldsymbol{i} + (-3+2x)\,\boldsymbol{k} - (2z\boldsymbol{i} - 3\boldsymbol{k}) = 4\boldsymbol{i} + 2x\boldsymbol{k}.$$

Next,

$$\nabla^2\boldsymbol{A} = \left(\nabla^2 A_1, \nabla^2 A_2, \nabla^2 A_3\right).$$

We have

$$\begin{aligned}
\nabla^2 A_1 &= \frac{\partial^2 A_1}{\partial x^2} + \frac{\partial^2 A_1}{\partial y^2} + \frac{\partial^2 A_1}{\partial z^2} \\
&= \frac{\partial^2}{\partial x^2}\left(2x^2\right) + \frac{\partial^2}{\partial y^2}\left(2x^2\right) + \frac{\partial^2}{\partial z^2}\left(2x^2\right) = \frac{\partial}{\partial x}(4x) = 4; \\
\nabla^2 A_2 &= \frac{\partial^2 A_2}{\partial x^2} + \frac{\partial^2 A_2}{\partial y^2} + \frac{\partial^2 A_2}{\partial z^2} \\
&= \frac{\partial^2}{\partial x^2}(-3yz) + \frac{\partial^2}{\partial y^2}(-3yz) + \frac{\partial^2}{\partial z^2}(-3yz) = \frac{\partial}{\partial y}(-3z) + \frac{\partial}{\partial z}(-3y) = 0; \\
\nabla^2 A_3 &= \frac{\partial^2 A_3}{\partial x^2} + \frac{\partial^2 A_3}{\partial y^2} + \frac{\partial^2 A_3}{\partial z^2} \\
&= \frac{\partial^2}{\partial x^2}\left(xz^2\right) + \frac{\partial^2}{\partial y^2}\left(xz^2\right) + \frac{\partial^2}{\partial z^2}\left(xz^2\right) = \frac{\partial}{\partial x}\left(z^2\right) + \frac{\partial}{\partial z}(2xz) = 2x.
\end{aligned}$$

Therefore

$$\nabla^2\boldsymbol{A} = 4\boldsymbol{i} + 2x\boldsymbol{k}.$$

(10 marks)

4 (i) On C, $x = 3\cos\theta$ and $y = 3\sin\theta$ for $0 \le \theta \le 2\pi$ so $\boldsymbol{r} = (3\cos\theta, 3\sin\theta, 0)$. Hence

$$\boldsymbol{dr} = (-3\sin\theta, 3\cos\theta, 0)\ d\theta.$$

Therefore

$$\begin{aligned}
I &= \int_{\theta=0}^{2\pi} (9\sin\theta, 12\cos\theta, 0) \,\boldsymbol{.}\, (-3\sin\theta, 3\cos\theta, 0)\, d\theta \\
&= \int_{\theta=0}^{2\pi} \left(-27\sin^2\theta + 36\cos^2\theta\right) d\theta \\
&= \int_{\theta=0}^{2\pi} \left(63\cos^2\theta - 27\right) d\theta \\
&= \int_{\theta=0}^{2\pi} \left(\frac{63\left(\cos\left(2\theta\right)+1\right)}{2} - 27\right) d\theta \\
&= \int_{\theta=0}^{2\pi} \frac{63\cos\left(2\theta\right)+9}{2} d\theta \\
&= \left[\frac{63\sin\left(2\theta\right)}{4} + 9\frac{\theta}{2}\right]_{\theta=0}^{2\pi} = 9\pi.
\end{aligned}$$

$$\operatorname{curl}\boldsymbol{F} = \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ 3y & 4x & 2z^2 \end{vmatrix} = \boldsymbol{i}(0) - \boldsymbol{j}(0) + \boldsymbol{k}(1) = \boldsymbol{k}.$$

On S_1, we have $r = 3$ and therefore the co-ordinates are:

$$x = 3\sin\theta\cos\phi, \qquad y = 3\sin\theta\sin\phi, \qquad z = 3\cos\theta.$$

Also, since $r = 3$,

$$dS = r^2\sin\theta\, d\theta\, d\phi = 9\sin\theta\, d\phi\, d\theta.$$

The unit normal $\hat{\boldsymbol{n}}$ to the surface S_1 is

$$\hat{\boldsymbol{n}} = \left(\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta\right),$$

and therefore

$$\boldsymbol{dS} = \hat{\boldsymbol{n}}\, dS = \left(9\sin\theta\cos\phi, 9\sin\theta\sin\phi, 9\cos\theta\right)\ \sin\theta\, d\phi\, d\theta.$$

 Question 4 continued on next page

So

$$\text{curl}\,\boldsymbol{F}.\boldsymbol{dS} = 9\cos\theta\sin\theta\,d\phi\,d\theta.$$

and

$$\begin{aligned} J &= \int_{\theta=0}^{\frac{\pi}{2}}\int_{\phi=0}^{2\pi} 9\cos\theta\sin\theta\,d\phi\,d\theta \\ &= 18\pi\int_{\theta=0}^{\frac{\pi}{2}}\cos\theta\sin\theta\,d\theta \\ &= 18\pi\left[\frac{-\cos^2\theta}{2}\right]_{\theta=0}^{\frac{\pi}{2}} \\ &= 18\pi\frac{1}{2} = 9\pi = I. \end{aligned}$$

(16 marks)

(ii) We have

$$\begin{aligned} \text{div}\,\boldsymbol{F} &= \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z} \\ &= \frac{\partial}{\partial x}(3y) + \frac{\partial}{\partial y}(4x) + \frac{\partial}{\partial z}\left(2z^2\right) \\ &= 0 + 0 + 4z = 4z. \end{aligned}$$

To describe the volume, use spherical polar co-ordinates so that

$$x = r\sin\theta\cos\phi, \qquad y = r\sin\theta\sin\phi, \qquad z = r\cos\theta.$$

Then $dV = r^2\sin\theta\,dr\,d\theta\,d\phi$, and the volume integral is

$$\begin{aligned} K &= \int_{r=0}^{3}\int_{\theta=0}^{\frac{\pi}{2}}\int_{\phi=0}^{2\pi} 4r^3\sin\theta\cos\theta\,d\phi\,d\theta\,dr \\ &= 2\pi\int_{r=0}^{3}\int_{\theta=0}^{\frac{\pi}{2}} 4r^3\sin\theta\cos\theta\,d\theta\,dr \\ &= 2\pi\int_{r=0}^{3}\left[2r^3\sin^2\theta\right]_{\theta=0}^{\frac{\pi}{2}}\,dr \\ &= 4\pi\int_{r=0}^{3} r^3\,dr = 4\pi\left[\frac{1}{4}r^4\right]_{r=0}^{3} = 81\pi. \end{aligned}$$

By Gauss' Divergence Theorem

$$L = \iint_S \boldsymbol{F}.\boldsymbol{dS} = K = 81\pi.$$

(9 marks)

End of Question Paper