Data provided: Formula Sheet

The University Of Sheffield.

MAS243

SCHOOL OF MATHEMATICS AND STATISTICS **Spring Semester 2010–2011**

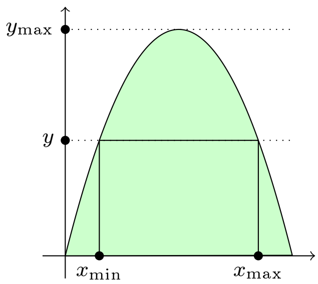
Mathematics IV (Electrical) 2 **hours**

Attempt all FOUR questions. Note that there are different numbers of marks for different questions.

1 (i) The function $f(x, y) = y^2 + 3x^4 - 4x^3 - 12x^2$ has precisely three critical points, precisely one of which is a global minimum. Find and classify the critical points, and thus find the minimum value of $f(x, y)$. ***(13 marks)***

(ii) Use the method of Lagrange multipliers to find the maximum and minimum values of $x - y$ subject to the constraint $x^2 + y^2 = x + y$. ***(12 marks)***

2 Consider the following region D, where the upper curve has equation $y = 4x(1-x)$.



(i) Find $y_{\max}$. ***(3 marks)***

(ii) Find $x_{\min}$ and $x_{\max}$ in terms of y. ***(3 marks)***

(iii) Now consider the integral

$$I = \iint_D \sqrt{\frac{y}{1-y}}\, dA$$

Work out the limits to give two different expressions for I, one as an integral of the form $\int_{x=\cdots}^{\cdots} \int_{y=\cdots}^{\cdots} \sqrt{\frac{y}{1-y}}\, dy\, dx$, and the other of the form $\int_{y=\cdots}^{\cdots} \int_{x=\cdots}^{\cdots} \sqrt{\frac{y}{1-y}}\, dx\, dy$. ***(8 marks)***

(iv) Use the second expression to evaluate I. ***(6 marks)***

3 (i) Consider the vector field $\mathbf{u} = (-x^2y - y^3, x^3 + xy^2, z^3)$. Calculate $\nabla.\mathbf{u}$, $\nabla(\nabla.\mathbf{u})$, $\nabla\times\mathbf{u}$, $\nabla\times(\nabla\times\mathbf{u})$ and $\nabla^2(\mathbf{u})$. Verify that $\nabla\times(\nabla\times\mathbf{u}) = \nabla(\nabla.\mathbf{u}) - \nabla^2(\mathbf{u})$. ***(16 marks)***

(ii) Consider the scalar field $f(x, y, z) = e^{-x^2-y^2-z^2}$. Find $\nabla(f)$ and $\nabla^2(f)$. Give a geometric description of the points where $\nabla^2(f) = 0$. ***(9 marks)***

4 (i) Let S be the surface given by $z = (x^2 + y^2)/100$ with $0 \leq z \leq 1$, and let C be the boundary of S. Consider the vector field $\mathbf{F} = z\,\mathbf{i} + x\,\mathbf{j} + y\,\mathbf{k}$. Evaluate the integrals $\iint_S (\nabla\times\mathbf{F}).d\mathbf{A}$ and $\int_C \mathbf{F}.d\mathbf{r}$ separately, and check that they are the same (in accordance with Stokes's Theorem). ***(18 marks)***

(ii) Let E be the spherical ball of radius one centred at the origin, and let T be the boundary of E. Let $\mathbf{G}$ be the vector field $(0, 0, z)$. Evaluate the integrals $\iiint_E \nabla.\mathbf{G}\,dV$ and $\iint_T \mathbf{G}.d\mathbf{A}$ separately, and check that they are the same (in accordance with the Divergence Theorem). ***(12 marks)***

You may use the identity

$$\sin(\alpha)\cos^2(\alpha) = \frac{1}{4}(\sin(3\alpha) + \sin(\alpha)).$$

End of Question Paper

Formula Sheet for MAS243
Trigonometry

$$\cos(A \pm B) = \cos A \cos B \mp \sin A \sin B$$

$$\sin(A \pm B) = \sin A \cos B \pm \cos A \sin B$$

$$\tan(A \pm B) = \frac{\tan A \pm \tan B}{1 \mp \tan A \tan B}$$

$$a \cos\theta + b \sin\theta = R\cos(\theta - \alpha) \text{ where } R = \sqrt{a^2 + b^2} \text{ and } \cos\alpha = \frac{a}{R},\ \sin\alpha = \frac{b}{R}$$

$$\cos^2\theta = \frac{1}{2}(\cos 2\theta + 1)$$

$$\cos^3\theta = \frac{1}{4}(3\cos\theta + \cos 3\theta)$$

$$\cos^4\theta = \frac{1}{8}(3 + 4\cos 2\theta + \cos 4\theta)$$

$$\sin^2\theta = \frac{1}{2}(1 - \cos 2\theta)$$

$$\sin^3\theta = \frac{1}{4}(3\sin\theta - \sin 3\theta)$$

$$\sin^4\theta = \frac{1}{8}(3 - 4\cos 2\theta + \cos 4\theta)$$