MAS243 Exam 2010/11

(1)

(i) The function $f(x,y) = y^2 + 3x^4 - 4x^3 - 12x^2$ has precisely three critical points, precisely one of which is a global minimum. Find and classify the critical points, and thus find the minimum value of $f(x,y)$. **(13 marks)**

(ii) Use the method of Lagrange multipliers to find the maximum and minimum values of $x - y$ subject to the constraint $x^2 + y^2 = x + y$. **(12 marks)**

Solution:

(i) The critical points are the points where both of the following partial derivatives are zero.

$$f_x(x,y) = 12x^3 - 12x^2 - 24x\textbf{[1]} = 12x(x^2 - x - 2) = 12x(x+1)(x-2)$$
$$f_y(x,y) = 2y.\textbf{[1]}$$

This means that $y = 0$,[1]and x is -1, 0 or 2,[1]so the critical points are at $(-1,0)$, $(0,0)$ and $(2,0)$.[1]To classify them, we need the Hessian matrix of second-order partial derivatives:

$$f_{xx}(x,y) = 36x^2 - 24x - 24 = 12(3x^2 - 2x - 2)\textbf{[1]}$$
$$f_{xy}(x,y) = f_{yx}(x,y) = 0$$
$$f_{yy}(x,y) = 2\textbf{[1]}$$
$$H = \begin{bmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{bmatrix} = \begin{bmatrix} 12(3x^2 - 2x - 2) & 0 \\ 0 & 2 \end{bmatrix}.$$

The key numbers to consider are $A_1 = f_{xx}$ and $A_2 = \det(H)$, which in this case is just $2f_{xx}$[1]. If $f_{xx} > 0$ then A_1 and A_2 are both positive so we have a local minimum. If $f_{xx} < 0$ then A_1 and A_2 are both negative so we have a saddle point. We can check through our list of critical points as follows:

$$f_{xx}(-1,0) = 36 + 24 - 24 = 36 > 0$$
$$f_{xx}(0,0) = -24 < 0$$
$$f_{xx}(2,0) = 144 - 48 - 24 = 72 > 0$$

so $(-1,0)$ and $(2,0)$ are local minima [2], whereas $(0,0)$ is a saddle [1]. We are told that there is a global minimum, so it must be one of the local minima. The values of f there are

$$f(-1,0) = 0 + 3 + 4 - 12 = -5$$
$$f(2,0) = 0 + 3 \times 16 - 4 \times 8 - 12 \times 4 = -32,$$

so we see that the global minimum is -32, attained at $(2,0)$[2].

(ii) We need to find the unconstrained critical points of the function $L = x - y - \lambda(x^2 + y^2 - x - y)$ [1]. These are the points where the following three equations hold: [3]

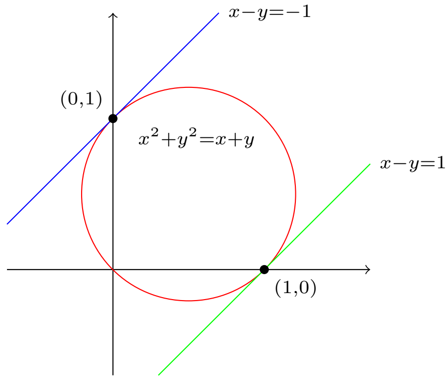
$$L_\lambda = x + y - x^2 - y^2 = 0 \qquad \text{(A)}$$
$$L_x = 1 - 2\lambda x + \lambda = 0 \qquad \text{(B)}$$
$$L_y = -1 - 2\lambda y + \lambda = 0. \qquad \text{(C)}$$

We can rearrange (B) and (C) to get $\lambda = -1/(1-2x) = 1/(1-2y)$, which in turn gives $1-2y = 2x-1$ or $y = 1-x$ **[2]**. Substituting this in (A) gives

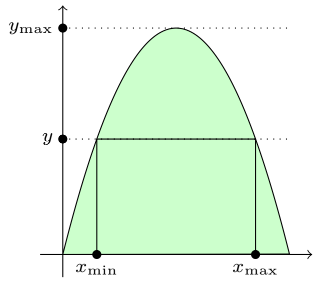
$$0 = x + y - x^2 - y^2 = x + (1-x) - x^2 - (1-x)^2 = 1 - x^2 - 1 + 2x - x^2 = 2x(1-x),$$

so $x = 0$ or $x = 1$ **[2]**. As $y = 1-x$, we see that the constrained critical points are at $(0,1)$ (where $x - y = -1$) and $(1,0)$ (where $x - y = +1$) **[2]**. It follows that the minimum value of $x - y$ is -1, and the maximum is $+1$ **[2]**.

The diagram (which students are not required to draw) is as follows:



(2) Consider the following region D, where the upper curve has equation $y = 4x(1-x)$.



(i) Find $y_{\max}$. **(3 marks)**

(ii) Find $x_{\min}$ and $x_{\max}$ in terms of y. **(3 marks)**

(iii) Now consider the integral

$$I = \iint_D \sqrt{\frac{y}{1-y}}\, dA$$

Work out the limits to give two different expressions for I, one as an integral of the form $\int_{x=\cdots}^{\cdots}\int_{y=\cdots}^{\cdots}\sqrt{\frac{y}{1-y}}\,dy\,dx$, and the other of the form $\int_{y=\cdots}^{\cdots}\int_{x=\cdots}^{\cdots}\sqrt{\frac{y}{1-y}}\,dx\,dy$. **(8 marks)**

(iv) Use the second expression to evaluate I. **(6 marks)**

Solution:

(i) The maximum value of y occurs at a point where $dy/dx = 0$ [1]. We have $y = 4x - 4x^2$, so $dy/dx = 4 - 8x$, and this is only zero at $x = 1/2$ [1]. At that point we have $y = 4x(1-x) = 4.\frac{1}{2}.\frac{1}{2} = 1$. Thus, $y_{\max} = 1$ [1].

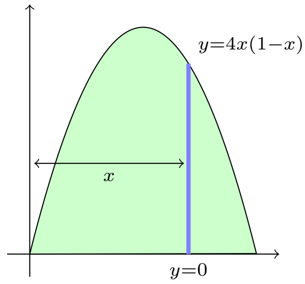
(ii) From the picture it is clear that $x_{\min}$ and $x_{\max}$ are the two values of x where $4x(1-x) = y$, or equivalently $4x^2 - 4x + y = 0$ [1]. By the standard quadratic formula, we have

$$x = \frac{4 \pm \sqrt{4^2 - 4 \times 4 \times y}}{2 \times 4} = \frac{4 \pm 4\sqrt{1-y}}{8} = \tfrac{1}{2}(1 \pm \sqrt{1-y}).\text{[1]}$$

This means that $x_{\min} = \frac{1}{2}(1 - \sqrt{1-y})$ and $x_{\max} = \frac{1}{2}(1 + \sqrt{1-y})$ [1].

(Note that the minimum and maximum values of x over the whole diagram are 0 and 1. However, it is clear from the diagram and the instruction "Find $x_{\min}$ and $x_{\max}$ *in terms of* y" that this is not the required answer. You need to find the minimum and maximum values on the horizontal strip at height y, not on the whole region D.)

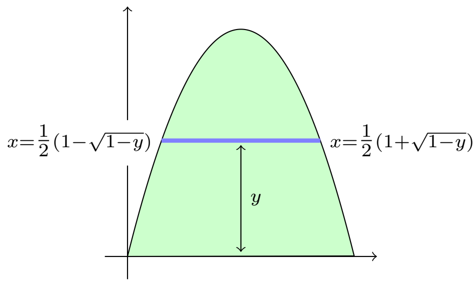
(iii) For the first expression, the overall limits of x are $0 \leq x \leq 1$ [1]. For a fixed value of x, the limits of y are $y = 0$ (at the bottom of the indicated strip) and $y = 4x(1-x)$ (at the top) [2].



We thus have

$$I = \int_{x=0}^{1} \int_{y=0}^{4x(1-x)} \sqrt{\frac{y}{1-y}}\, dy\, dx.\textbf{[1]}$$

For the second expression, the overall limits of y are $0 \le y \le y_{\max} = 1$ **[1]**. For a fixed value of y, the variable x ranges from $x = x_{\min} = \frac{1}{2}(1 - \sqrt{1-y})$ (at the left hand end of the indicated strip) to $x = x_{\max} = \frac{1}{2}(1 + \sqrt{1-y})$ (at the right hand end) **[2]**.



We therefore have

$$I = \int_{y=0}^{1} \int_{x=\frac{1}{2}(1-\sqrt{1-y})}^{\frac{1}{2}(1+\sqrt{1-y})} \sqrt{\frac{y}{1-y}}\, dx\, dy.\textbf{[1]}$$

(iv) We will use the second expression for I. For the inner integral, we just have

$$\begin{aligned}
\int_{x=\frac{1}{2}(1-\sqrt{1-y})}^{\frac{1}{2}(1+\sqrt{1-y})} \sqrt{\frac{y}{1-y}}\, dx &= \left[\sqrt{\frac{y}{1-y}}\, x\right]_{x=\frac{1}{2}(1-\sqrt{1-y})}^{\frac{1}{2}(1+\sqrt{1-y})} \textbf{[1]} \\
&= \frac{1}{2}\sqrt{\frac{y}{1-y}}\left(1 + \sqrt{1-y} - (1 - \sqrt{1-y})\right) \\
&= \frac{1}{2}\sqrt{\frac{y}{1-y}}.2\sqrt{1-y}\textbf{[2]} \\
&= \sqrt{y} = y^{\frac{1}{2}}\textbf{[1]}.
\end{aligned}$$

Thus, the outer integral is

$$I = \int_{y=0}^{1} y^{\frac{1}{2}}\, dy = \left[\tfrac{2}{3} y^{\frac{3}{2}}\right]_{y=0}^{1} = \frac{2}{3}\textbf{[2]}.$$

(3)

(i) Consider the vector field $\mathbf{u} = (-x^2y - y^3, x^3 + xy^2, z^3)$. Calculate $\nabla.\mathbf{u}$, $\nabla(\nabla.\mathbf{u})$, $\nabla \times \mathbf{u}$, $\nabla \times (\nabla \times \mathbf{u})$ and $\nabla^2(\mathbf{u})$. Verify that $\nabla \times (\nabla \times \mathbf{u}) = \nabla(\nabla.\mathbf{u}) - \nabla^2(\mathbf{u})$. **(16 marks)**

(ii) Consider the scalar field $f(x, y, z) = e^{-x^2-y^2-z^2}$. Find $\nabla(f)$ and $\nabla^2(f)$. Give a geometric description of the points where $\nabla^2(f) = 0$. **(9 marks)**

Solution:

(i)

$$\begin{aligned}
\nabla.\mathbf{u} &= \frac{\partial}{\partial x}(-x^2y - y^3) + \frac{\partial}{\partial y}(x^3 + xy^2) + \frac{\partial}{\partial z}(z^3) \\
&= -2xy + 2xy + 3z^2 = 3z^2 \text{[2]} \\
\nabla(\nabla.\mathbf{u}) &= \nabla(3z^2) = \left(\frac{\partial}{\partial x}(3z^2),\ \frac{\partial}{\partial y}(3z^2),\ \frac{\partial}{\partial z}(3z^2)\right) \\
&= (0, 0, 6z)\text{[2]} \\
\nabla \times \mathbf{u} &= \det \begin{bmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ -x^2y - y^3 & x^3 + xy^2 & z^3 \end{bmatrix} \text{[1]} \\
&= \left((z^3)_y - (x^3 + xy^2)_z,\ (-x^2y - y^3)_z - (z^3)_x,\ (x^3 + xy^2)_x - (-x^2y - y^3)_y\right) \\
&= (0, 0, (3x^2 + y^2) - (-x^2 - 3y^2)) = (0, 0, 4(x^2 + y^2))\text{[2]} \\
\nabla \times (\nabla \times \mathbf{u}) &= \nabla \times (0, 0, 4(x^2 + y^2)) \\
&= \det \begin{bmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 0 & 0 & 4x^2 + 4y^2 \end{bmatrix} \text{[1]} \\
&= \left(\frac{\partial}{\partial y}(4x^2 + 4y^2),\ -\frac{\partial}{\partial x}(4x^2 + 4y^2),\ 0\right) \\
&= (8y, -8x, 0)\text{[2]} \\
\nabla^2(-x^2y - y^3) &= (-x^2y - y^3)_{xx} + (-x^2y - y^3)_{yy} + (-x^2y - y^3)_{zz} \\
&= -2y + (-6y) + 0 = -8y\text{[2]} \\
\nabla^2(x^3 + xy^2) &= (x^3 + xy^2)_{xx} + (x^3 + xy^2)_{yy} + (x^3 + xy^2)_{zz} \\
&= 6x + 2x + 0 = 8x\text{[1]} \\
\nabla^2(z^3) &= (z^3)_{xx} + (z^3)_{yy} + (z^3)_{zz} \\
&= 0 + 0 + 6z = 6z\text{[1]} \\
\nabla^2(\mathbf{u}) &= (\nabla^2(-x^2y - y^3),\ \nabla^2(x^3 + xy^2),\ \nabla^2(z^3)) \\
&= (-8y,\ 8x,\ 6z)\text{[1]} \\
\nabla(\nabla.\mathbf{u}) - \nabla^2(\mathbf{u}) &= (0, 0, 6z) - (-8y, 8x, 6z) = (8y, -8x, 0) \\
&= \nabla \times (\nabla \times \mathbf{u}).\text{[1]}
\end{aligned}$$

(ii) First, the chain rule gives

$$f_x = e^{-x^2-y^2-z^2}\frac{\partial}{\partial x}(-x^2 - y^2 - z^2) = -2x\, e^{-x^2-y^2-z^2} = -2x\, f\text{[1]}$$

and similarly $f_y = -2y\, f$ and $f_z = -2z\, f$, [1]so

$$\nabla(f) = (-2x\, f,\ -2y\, f,\ -2z\, f) = -2e^{-x^2-y^2-z^2}(x,\ y,\ z).\text{[1]}$$

Next, we have

$$f_{xx} = \frac{\partial}{\partial x}(-2x\, f) = -2f - 2xf_x = -2f + 4x^2 f = (4x^2 - 2)f.[1]$$

Similarly, we have $f_{yy} = (4y^2 - 2)f$ and $f_{zz} = (4z^2 - 2)f$.[1]It follows that

$$\begin{aligned}\nabla^2(f) &= f_{xx} + f_{yy} + f_{zz}[1] \\ &= (4x^2 - 2)f + (4y^2 - 2)f + (4z^2 - 2)f \\ &= (4(x^2 + y^2 + z^2) - 6)e^{-x^2-y^2-z^2}[1].\end{aligned}$$

In particular, $\nabla^2(f) = 0$ when $4(x^2+y^2+z^2) = 6$, which means that $\sqrt{x^2 + y^2 + z^2} = \sqrt{3/2}$ [1]. This describes a sphere of radius $\sqrt{3/2}$ centred at the origin [1].

(4)

(i) Let S be the surface given by $z = (x^2 + y^2)/100$ with $0 \leq z \leq 1$, and let C be the boundary of S. Consider the vector field $\mathbf{F} = z\,\mathbf{i} + x\,\mathbf{j} + y\,\mathbf{k}$. Evaluate the integrals $\iint_S (\nabla \times \mathbf{F}).d\mathbf{A}$ and $\int_C \mathbf{F}.d\mathbf{r}$ separately, and check that they are the same (in accordance with Stokes's Theorem). **(18 marks)**

(ii) Let E be the spherical ball of radius one centred at the origin, and let T be the boundary of E. Let $\mathbf{G}$ be the vector field $(0, 0, z)$. Evaluate the integrals $\iiint_E \nabla.\mathbf{G}\,dV$ and $\iint_T \mathbf{G}.d\mathbf{A}$ separately, and check that they are the same (in accordance with the Divergence Theorem). **(12 marks)**

You may use the identity

$$\sin(\alpha)\cos^2(\alpha) = \frac{1}{4}(\sin(3\alpha) + \sin(\alpha)).$$

Solution:

(i) First, we have

$$\nabla \times \mathbf{F} = \det \begin{bmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ z & x & y \end{bmatrix} = (1-0,\ 1-0,\ 1-0) = (1,1,1)[2].$$

Next, for a surface $z = f(x,y)$ we have $d\mathbf{A} = (-f_x, -f_y, 1)dx\,dy$. In our case the relevant function is $f = (x^2+y^2)/100$, so $d\mathbf{A} = (-x/50, -y/50, 1)dx\,dy$ so $(\nabla \times \mathbf{F}).d\mathbf{A} = (1-(x+y)/50)dx\,dy$ [2]. This can be rewritten in cylindrical polar coordinates as

$$(\nabla \times \mathbf{F}).d\mathbf{A} = (1 - r(\cos(\theta) + \sin(\theta))/50)r\,d\theta\,dr.[2]$$

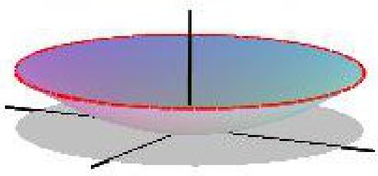
The limits $0 \leq z = (x^2+y^2)/100 = r^2/100 \leq 1$ translate to $0 \leq r \leq 10$ with $0 \leq \theta \leq 2\pi$ [2]. We thus have

$$\begin{aligned}\iint_S (\nabla \times \mathbf{F}).d\mathbf{A} &= \int_{r=0}^{10}\int_{\theta=0}^{2\pi} (1 - r(\cos(\theta)+\sin(\theta))/50)r\,d\theta\,dr[1] \\ &= \int_{r=0}^{10} \Big[r\theta - r^2(\sin(\theta) - \cos(\theta))/50\Big]_{\theta=0}^{2\pi} dr[1] \\ &= \int_{r=0}^{10} 2\pi r\,dr = \Big[\pi r^2\Big]_{r=0}^{10} = 100\pi.[1]\end{aligned}$$

On the other hand, the boundary curve C has $z = 1$ and $r = 10$ so it can be parametrised as $\mathbf{r} = (x,y,z) = (10\cos(\theta), 10\sin(\theta), 1)$ for $0 \leq \theta \leq 2\pi$ [2]. This gives

$$\begin{aligned} d\mathbf{r} &= (-10\sin(\theta),\ 10\cos(\theta),\ 0)\,d\theta[1] \\ \mathbf{F} &= (z,x,y) = (1,\ 10\cos(\theta),\ 10\sin(\theta))[1] \\ \mathbf{F}.d\mathbf{r} &= (-10\sin(\theta) + 100\cos^2(\theta))d\theta = (-10\sin(\theta) + 50 + 50\cos(2\theta))d\theta[2] \\ \int_C \mathbf{F}.d\mathbf{r} &= \int_{\theta=0}^{2\pi} (-10\sin(\theta) + 50 + 50\cos(2\theta))d\theta = \Big[10\cos(\theta) + 50\theta + 25\sin(2\theta)\Big]_{\theta=0}^{2\pi} \\ &= (10 + 100\pi + 0) - (10 + 0 + 0) = 100\pi[1].\end{aligned}$$

The picture (which students are not required to draw) is as follows:



The curved surface is S, and the flat disc below it is the disc of radius 10 centred at the origin in the xy-plane. The circle C is the edge of the curved surface; it is shown in red. The vertical scale has been exaggerated for clarity.

(ii) First, we have $\nabla.\mathbf{G} = \frac{\partial 0}{\partial x} + \frac{\partial 0}{\partial y} + \frac{\partial z}{\partial z} = 0 + 0 + 1 = 1$ [2], so

$$\iiint_E \nabla.\mathbf{G}\, dV = \iiint_E 1\, dV = \text{volume}(E) = 4\pi/3.$$

Explicitly, we have

$$\begin{aligned}
\iiint_E 1\, dV &= \int_{r=0}^{1} \int_{\theta=0}^{2\pi} \int_{\phi=0}^{\pi} r^2 \sin(\phi)\, d\phi\, d\theta\, dr \\
&= \left(\int_{r=0}^{1} r^2\, dr\right) \left(\int_{\theta=0}^{2\pi} 1\, d\theta\right) \left(\int_{\phi=0}^{\pi} \sin(\phi)\, d\phi\right) \text{[1]} \\
&= \left[\frac{r^3}{3}\right]_{r=0}^{1} \left[\theta\right]_{\theta=0}^{2\pi} \left[-\cos(\phi)\right]_{\phi=0}^{\pi} \quad \text{[1]} \\
&= \frac{1}{3} \times 2\pi \times (1-(-1)) = \frac{4\pi}{3}.\text{[1]}
\end{aligned}$$

Next, it is standard that on the unit sphere we have

$$d\mathbf{A} = \mathbf{r}\, dA = \sin(\phi)\mathbf{r} d\theta\, d\phi = (\sin^2(\phi)\cos(\theta),\ \sin^2(\phi)\sin(\theta),\ \sin(\phi)\cos(\phi))\, d\theta\, d\phi.\text{[1]}$$

On the unit sphere we also have

$$\mathbf{G} = (0, 0, z) = (0, 0, \cos(\phi))\text{[1]}$$

so $\mathbf{G}.d\mathbf{A} = \sin(\phi)\cos^2(\phi)\, d\theta\, d\phi$. Using this together with the hint we get

$$\begin{aligned}
\iint_T \mathbf{G}.d\mathbf{A} &= \int_{\phi=0}^{\pi} \int_{\theta=0}^{2\pi} \sin(\phi)\cos^2(\phi)\, d\theta\, d\phi\text{[1]} \\
&= 2\pi \int_{\phi=0}^{\pi} \sin(\phi)\cos^2(\phi)\, d\phi\text{[1]} \\
&= \frac{2\pi}{4} \int_{\phi=0}^{\pi} \sin(3\phi) + \sin(\phi)\, d\phi\text{[1]} \\
&= \frac{\pi}{2} \left[-\frac{1}{3}\cos(3\phi) - \cos(\phi)\right]_{\phi=0}^{\pi} \quad \text{[1]} \\
&= \frac{\pi}{2}\left((\tfrac{1}{3}+1) - (-\tfrac{1}{3}-1)\right) = \frac{4\pi}{3}\text{[1]}.
\end{aligned}$$