MAS243 Exam 2011/12

(1)

(i) The function $f(x,y) = (2+\cos(x))(2+\sin(y))$ has four critical points with $0 \leq x, y < 2\pi$. Find and classify these critical points, and determine the maximum and minimum values of f. **(13 marks)**

(ii) Use the method of Lagrange multipliers to find the maximum and minimum values of the function $x^2 + 4xy + 4y^2$ on the circle of radius $\sqrt{5}$ centred at the origin. **(12 marks)**

Solution:

(i) We first find the relevant partial derivatives:

$$\begin{aligned} f_x(x,y) &= -\sin(x)(2+\sin(y)) \\ f_y(x,y) &= (2+\cos(x))\cos(y) \\ f_{xx}(x,y) &= -\cos(x)(2+\sin(y)) \\ f_{xy}(x,y) &= -\sin(x)\cos(y) \\ f_{yy}(x,y) &= -(2+\cos(x))\sin(y). \text{[4]} \end{aligned}$$

The critical points are the points where $f_x = f_y = 0$. As $\cos(x)$ and $\sin(y)$ lie between -1 and 1 we see that $2+\sin(y)$ and $2+\cos(x)$ can never be zero. The critical points therefore occur where $\sin(x) = \cos(y) = 0$ [1]. We are told to look for critical points where $0 \leq x, y < 2\pi$, so we must have $x = 0$ or $x = \pi$, and $y = \pi/2$ or $y = 3\pi/2$ [2]. We can classify these using the Hessian:

$$H = \begin{bmatrix} f_{xx} & f_{xy} \\ f_{xy} & f_{yy} \end{bmatrix} = \begin{bmatrix} -\cos(x)(2+\sin(y)) & -\sin(x)\cos(y) \\ -\sin(x)\cos(y) & -(2+\cos(x))\sin(y) \end{bmatrix}. \text{[1]}$$

Recall that A_1 is the top left entry in the Hessian, and A_2 is the determinant. If $A_2 < 0$ at a critical point, then it is a saddle. If $A_2 > 0$ and $A_1 > 0$ then it is a local minimum, and if $A_2 > 0$ and $A_1 < 0$ it is a local maximum. We can work everything out at our four critical points as follows:

(x,y)	$\cos(x)$	$\sin(y)$	H	A_1	A_2	type	$f(x,y)$
$(0,\pi/2)$	1	1	$\left[\begin{smallmatrix} -3 & 0 \\ 0 & -3 \end{smallmatrix}\right]$	-3	9	local maximum [1]	9
$(0,3\pi/2)$	1	-1	$\left[\begin{smallmatrix} -1 & 0 \\ 0 & 3 \end{smallmatrix}\right]$	-1	-3	saddle [1]	3
$(\pi,\pi/2)$	-1	1	$\left[\begin{smallmatrix} 3 & 0 \\ 0 & -1 \end{smallmatrix}\right]$	3	-3	saddle [1]	3
$(\pi,3\pi/2)$	-1	-1	$\left[\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right]$	1	1	local minimum [1]	1

We now see that the minimum value is 1 and the maximum value is 9. [1]

(ii) The relevant circle has equation $x^2 + y^2 - 5 = 0$ [1], so we need to find the unconstrained critical points of the function

$$L(\lambda, x, y) = x^2 + 4xy + 4y^2 - \lambda(x^2 + y^2 - 5). \text{[1]}$$

The relevant partial derivatives are as follows:

$$L_\lambda = 5 - x^2 - y^2 [1]$$
$$L_x = 2x + 4y - 2x\lambda [1]$$
$$L_y = 4x + 8y - 2y\lambda [1].$$

Thus, the critical points are given by the following equations:

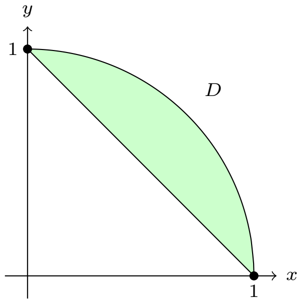
$$x^2 + y^2 = 5 \quad \text{(A)}$$
$$2x + 4y = 2x\lambda \quad \text{(B)}$$
$$4x + 8y = 2y\lambda. \quad \text{(C)}$$

If we multiply (B) by y we get $2xy + 4y^2 = 2xy\lambda$. If we multiply (C) by x we get $4x^2 + 8xy = 2xy\lambda$. Subtracting these two equations gives

$$4x^2 + 6xy - 4y^2 = 0 \quad \text{(D)}$$

We can divide by $4y^2$ to get $(x/y)^2 - \frac{2}{3}(x/y) - 1 = 0$ and then use the quadratic formula to get $x/y = -2$ or $x/y = \frac{1}{2}$. Equivalently, we have either $x = -2y$ or $x = y/2$ [3]. In the first case equation (A) becomes $5y^2 = 5$, so $y = \pm 1$, giving critical points $P_1 = (-2, 1)$ and $P_2 = (2, -1)$ [1]. In the second case equation (A) becomes $5y^2/4 = 5$ so $y = \pm 2$, giving critical points $P_3 = (1, 2)$ and $P_4 = (-1, -2)$ [1]. The function $x^2 + 4xy + 4y^2$ takes the value 0 at P_1 and P_2, and 25 at P_3 and P_4. Thus, the minimum value is 0 and the maximum value is 25 [2].

(2) Let D be the following region:



(The outer part of the boundary is part of the circle of radius one centred at the origin, and the inner part is a straight line.)

(a) Find appropriate limits to express the integral $I = \iint_D x^2\,dA$ as a double integral in two different orders:

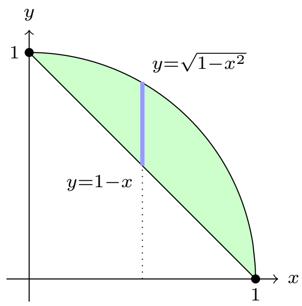
$$I = \int_{x=\cdots}^{\cdots}\int_{y=\cdots}^{\cdots} x^2\,dy\,dx = \int_{y=\cdots}^{\cdots}\int_{x=\cdots}^{\cdots} x^2\,dx\,dy.\textbf{(8 marks)}$$

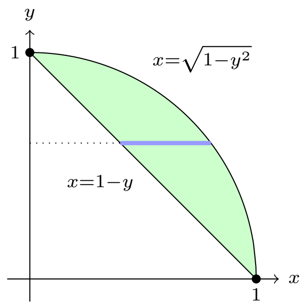
(b) Use a substitution to show that $\int_{t=0}^{1} t^2\sqrt{1-t^2}\,dt = \pi/16$. **(13 marks)**

(c) Use the first expression from (a) together with (b) to evaluate I. **(4 marks)**

Solution:

(a) It is clear that the overall limits for x are $0 \le x \le 1$ [1]. The equation of the inner boundary is $x + y = 1$, or equivalently $y = 1 - x$ [1]. The equation of the outer boundary is $x^2 + y^2 = 1$ [1], or equivalently $y = \sqrt{1-x^2}$ [1]. Thus, for any given value of x, we have $1 - x \le y \le \sqrt{1-x^2}$.





We conclude that

$$I = \int_{x=0}^{1}\int_{y=1-x}^{\sqrt{1-x^2}} x^2\,dy\,dx.[2]$$

By a similar argument (illustrated by the picture on the right) we have

$$I = \int_{y=0}^{1}\int_{x=1-y}^{\sqrt{1-y^2}} x^2\,dx\,dy.[2]$$

(b) We will evaluate the integral $J = \int_{t=0}^{1} t^2\sqrt{1-t^2}\,dt$ by substituting $t = \sin(\theta)$ [2]. The limits $t = 0$ and $t = 1$ become $\theta = 0$ and $\theta = \pi/2$ [1]. We have $dt/d\theta = \cos(\theta)$ so $dt = \cos(\theta)\,d\theta$ [2]. We also have

$$\sqrt{1-t^2} = \sqrt{1-\sin^2(\theta)} = \sqrt{\cos^2(\theta)} = \cos(\theta).[2]$$

Putting this together, and using some standard trigonometric identities, we get

$$\begin{aligned} J &= \int_{\theta=0}^{\pi/2} \sin^2(\theta)\cos(\theta)\ \cos(\theta)d\theta = \int_{\theta=0}^{\pi/2} (\sin(\theta)\cos(\theta))^2\ d\theta[1] \\ &= \frac{1}{4}\int_{\theta=0}^{\pi/2} \sin^2(2\theta)\ d\theta[2] = \frac{1}{8}\int_{\theta=0}^{\pi/2} 1-\cos(4\theta)\ d\theta[1] \\ &= \frac{1}{8}\left[\theta - \frac{1}{4}\sin(4\theta)\right]_{\theta=0}^{\pi/2} [1] = \frac{\pi}{16}.[1] \end{aligned}$$

(c) Using our first expression from part (a) we get

$$\begin{aligned} I &= \int_{x=0}^{1}\int_{y=1-x}^{\sqrt{1-x^2}} x^2\,dy\,dx = \int_{x=0}^{1} \left[x^2 y\right]_{y=1-x}^{\sqrt{1-x^2}} dx[1] \\ &= \int_{x=0}^{1} x^2\sqrt{1-x^2} - x^2 + x^3\,dx[1] = \frac{\pi}{16} + \left[\frac{x^4}{4} - \frac{x^3}{3}\right]_{x=0}^{1} [1] \\ &= \frac{\pi}{16} - \frac{1}{12}.[1] \end{aligned}$$

(3) Put $\mathbf{u} = (\cos(z),\ \sin(z),\ \exp(z))$ and $\mathbf{v} = (-\sin(z),\ \cos(z),\ \exp(-z))$.

(i) Simplify the following:

$$\begin{aligned} \mathbf{u}.\mathbf{v} \\ \mathbf{u}.\operatorname{curl}(\mathbf{u}) \\ \operatorname{curl}(\mathbf{u}) \times \operatorname{curl}(\mathbf{v}) \\ \operatorname{div}(\mathbf{u})\operatorname{div}(\mathbf{v}) \\ \mathbf{u} - \operatorname{curl}(\operatorname{curl}(\mathbf{u})) \\ \operatorname{div}(\operatorname{curl}(\operatorname{curl}(\operatorname{curl}(\mathbf{v})))) \end{aligned}$$

(18 marks)

(ii) Is there a function f such that $\mathbf{u} = \operatorname{grad}(f)$? Justify your answer. **(2 marks)**

(iii) Find a function g such that $\mathbf{u} = \operatorname{grad}(g) + \operatorname{curl}(\operatorname{curl}(\mathbf{u}))$ **(5 marks)**.

Solution:

(i)

$$\begin{aligned}
\mathbf{u}.\mathbf{v} &= -\cos(z)\sin(z) + \sin(z)\cos(z) + \exp(z)\exp(-z)[1] = \exp(z)\exp(-z) = 1[1] \\
\operatorname{curl}(\mathbf{u}) &= \det \begin{bmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \cos(z) & \sin(z) & \exp(z) \end{bmatrix} [1] \\
&= \mathbf{i}(0 - \cos(z)) - \mathbf{j}(0 - (-\sin(z))) + \mathbf{k}(0 - 0) \\
&= (-\cos(z),\ -\sin(z),\ 0)[1] \\
\mathbf{u}.\operatorname{curl}(\mathbf{u}) &= (\cos(z),\ \sin(z),\ \exp(z)).(-\cos(z),\ -\sin(z),\ 0) = -\cos^2(z) - \sin^2(z) = -1[1] \\
\operatorname{curl}(\mathbf{v}) &= \det \begin{bmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ -\sin(z) & \cos(z) & \exp(-z) \end{bmatrix} [1] \\
&= \mathbf{i}(0 - (-\sin(z))) - \mathbf{j}(0 - (-\cos(z))) + \mathbf{k}(0 - 0) \\
&= (\sin(z),\ -\cos(z),\ 0)[1] \\
\operatorname{curl}(\mathbf{u}) \times \operatorname{curl}(\mathbf{v}) &= \det \begin{bmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -\cos(z) & -\sin(z) & 0 \\ \sin(z) & -\cos(z) & 0 \end{bmatrix} [1] \\
&= \mathbf{i}(0-0) - \mathbf{j}(0-0) + \mathbf{k}(\cos^2(z) + \sin^2(z)) = (0,0,1)[1] \\
\operatorname{div}(\mathbf{u}) &= \frac{\partial}{\partial x}(\cos(z)) + \frac{\partial}{\partial y}(\sin(z)) + \frac{\partial}{\partial z}(\exp(z))[1] = 0 + 0 + \exp(z) = \exp(z)[1] \\
\operatorname{div}(\mathbf{v}) &= \frac{\partial}{\partial x}(-\sin(z)) + \frac{\partial}{\partial y}(\cos(z)) + \frac{\partial}{\partial z}(\exp(-z)) = 0 + 0 - \exp(-z) = -\exp(-z)[1] \\
\operatorname{div}(\mathbf{u})\operatorname{div}(\mathbf{v}) &= -\exp(z)\exp(-z) = -1[1] \\
\operatorname{curl}(\operatorname{curl}(\mathbf{u})) &= \det \begin{bmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ -\cos(z) & -\sin(z) & 0 \end{bmatrix} \\
&= \mathbf{i}(0 - (-\cos(z))) - \mathbf{j}(0 - \sin(z)) + \mathbf{k}(0 - 0) = (\cos(z),\ \sin(z),\ 0)[1] \\
\mathbf{u} - \operatorname{curl}(\operatorname{curl}(\mathbf{u})) &= (\cos(z),\ \sin(z),\ \exp(z)) - (\cos(z),\ \sin(z),\ 0) \\
&= (0,\ 0,\ e^z).[1]
\end{aligned}$$

For the last part, the correct approach is to say that $\text{div}(\text{curl}(\mathbf{w})) = 0$ for any vector field $\mathbf{w}$, and we can take $\mathbf{w} = \text{curl}(\text{curl}(\mathbf{v}))$ to get $\text{div}(\text{curl}(\text{curl}(\text{curl}(\mathbf{v})))) = 0$. **[3]**

Alternatively, we can calculate as follows:

$$\begin{aligned}
\text{curl}(\mathbf{v}) &= (\sin(z),\ -\cos(z),\ 0) \\
\text{curl}(\text{curl}(\mathbf{v})) &= (-\sin(z),\ \cos(z),\ 0) \\
\text{curl}(\text{curl}(\text{curl}(\mathbf{v}))) &= (\sin(z),\ -\cos(z),\ 0) \\
\text{div}(\text{curl}(\text{curl}(\text{curl}(\mathbf{v})))) &= 0 + 0 + 0 = 0.
\end{aligned}$$

(ii) We have seen that $\text{curl}(\mathbf{u}) = (-\cos(z),\ -\sin(z),\ 0) \neq 0$ so the vector field $\mathbf{u}$ cannot have a potential function. In other words, there is no function f such that $\text{grad}(f) = \mathbf{u}$ **[2]**.

(iii) We need to find g such that $\text{grad}(g) = \mathbf{u} - \text{curl}(\text{curl}(\mathbf{u})) = (0, 0, e^z)$ **[2]**. By inspection, the function $g = e^z$ has this property. In more detail, we must have $(g_x, g_y, g_z) = (0, 0, e^z)$. This gives $g_x = g_y = 0$, so g is a function of z only. We also have $g_z = e^z$, so $g = \int e^z\, dz = e^z$ **[3]**.

(4)

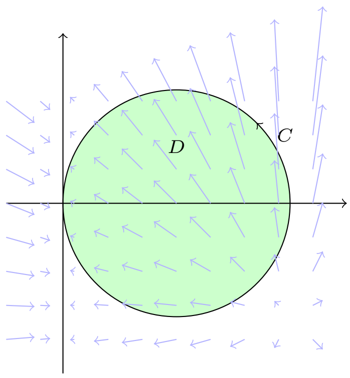
(a) Let C be the circle of radius one centred at $(1,0)$, and let $\mathbf{u}$ be the vector field $(x^2-2x, x+xy)$. Evaluate $\int_C \mathbf{u}.d\mathbf{r}$. **(6 marks)**

(b) Let $\mathbf{v}$ be the vector field (z,y,x), and let S be the sphere of radius 2 centred at the origin. Evaluate $\iint_S \mathbf{v}.d\mathbf{A}$ directly (without using the Divergence Theorem). **(14 marks)**

(c) Evaluate $\iiint_E xyz\,dV$, where E is the solid region given by $x,y,z \geq 0$ with $y \leq 1$ and $x+z \leq 1$. **(5 marks)**

Solution:

(a) The geometry is as follows:



We can parametrise C as $\mathbf{r} = (x,y) = (1+\cos(t), \sin(t))$ for $0 \leq t \leq 2\pi$ [1]. This gives $d\mathbf{r} = (-\sin(t), \cos(t))\,dt$ [1]and

$$\begin{aligned}
\mathbf{u} &= (x^2 - 2x,\ x + xy) \\
&= ((1+\cos(t))^2 - 2(1+\cos(t)), \quad (1+\cos(t)) + (1+\cos(t))\sin(t)) \\
&= (\cos^2(t) - 1, \quad 1 + \cos(t) + \sin(t) + \sin(t)\cos(t))[1] \\
\mathbf{u}.d\mathbf{r} &= -(\cos^2(t)-1)\sin(t) + (1+\cos(t)+\sin(t)+\sin(t)\cos(t))\cos(t) \\
&= \sin(t) - \cos^2(t)\sin(t) + \cos(t) + \cos^2(t) + \sin(t)\cos(t) + \sin(t)\cos^2(t) \\
&= \sin(t) + \cos(t) + \cos^2(t) + \sin(t)\cos(t)[1] \\
&= \sin(t) + \cos(t) + (1+\cos(2t))/2 + \sin(2t)/2[1] \\
\int_C \mathbf{u}.d\mathbf{r} &= \int_{t=0}^{2\pi} \sin(t) + \cos(t) + (1+\cos(2t))/2 + \sin(2t)/2 dt \\
&= \Big[-\cos(t) + \sin(t) + t/2 + \sin(2t)/4 - \cos(2t)/4 \Big]_{t=0}^{2\pi} = \pi.[1]
\end{aligned}$$

(b) Firstly, we can parametrise S as

$$(x,y,z) = (2\sin(\phi)\cos(\theta),\ 2\sin(\phi)\sin(\theta),\ 2\cos(\phi)), [2]$$

which gives

$$\begin{aligned}
\mathbf{v} &= (z,y,x) \\
&= (2\cos(\phi),\ 2\sin(\phi)\sin(\theta),\ 2\sin(\phi)\cos(\theta))[1]
\end{aligned}$$

We also have the following standard formulae for a sphere of radius 2:

$$\begin{aligned}\mathbf{n} &= (\sin(\phi)\cos(\theta),\ \sin(\phi)\sin(\theta),\ \cos(\phi))\\ dA &= 2^2\sin(\phi)d\theta\, d\phi = 4\sin(\phi)d\theta\, d\phi\\ d\mathbf{A} &= \mathbf{n}\, dA[2].\end{aligned}$$

Now

$$\begin{aligned}\mathbf{v}.\mathbf{n} &= 2\sin(\phi)\cos(\phi)\cos(\theta) + 2\sin^2(\phi)\sin^2(\theta) + 2\sin(\phi)\cos(\phi)\cos(\theta)\\ &= 4\sin(\phi)\cos(\phi)\cos(\theta) + 2\sin^2(\phi)\sin^2(\theta)[1]\\ \mathbf{v}.d\mathbf{A} &= 4\mathbf{v}.\mathbf{n}\sin(\phi)\, d\theta\, d\phi\\ &= \left(16\sin^2(\phi)\cos(\phi)\cos(\theta) + 8\sin^3(\phi)\sin^2(\theta)\right)\, d\theta\, d\phi[1]\\ \iint_S \mathbf{v}.d\mathbf{A} &= \int_{\phi=0}^{\pi}\int_{\theta=0}^{2\pi} 16\sin^2(\phi)\cos(\phi)\cos(\theta) + 8\sin^3(\phi)\sin^2(\theta)\, d\theta\, d\phi[1]\\ &= 16\left(\int_{\phi=0}^{\pi}\sin^2(\phi)\cos(\phi)\, d\phi\right)\left(\int_{\theta=0}^{2\pi}\cos(\theta)\, d\theta\right) +\\ &\qquad 8\left(\int_{\phi=0}^{\pi}\sin^3(\phi)\, d\phi\right)\left(\int_{\theta=0}^{2\pi}\sin^2(\theta)\, d\theta\right).[1]\end{aligned}$$

Here $\int_{\theta=0}^{2\pi}\cos(\theta)\, d\theta = \left[\sin(\theta)\right]_{\theta=0}^{\pi} = 0$ [1], so we need not evaluate $\int_{\phi=0}^{\pi}\sin^2(\phi)\cos(\phi)\, d\phi$. We also have the standard integral

$$\int_{\theta=0}^{2\pi}\sin^2(\theta)\, d\theta = \int_{\theta=0}^{2\pi}\tfrac{1}{2}(1-\cos(2\theta))\, d\theta = \left[\tfrac{1}{2}\theta - \frac{1}{4}\sin(2\theta)\right]_{\theta=0}^{2\pi} = \pi.[1]$$

The formula sheet tells us that $\sin^3(\phi) = \frac{3}{4}\sin(\phi) - \frac{1}{4}\sin(3\phi)$, which gives

$$\begin{aligned}\int_{\phi=0}^{\pi}\sin^3(\phi)\, d\phi &= \left[-\tfrac{3}{4}\cos(\phi) + \tfrac{1}{12}\cos(3\phi)\right]_{\phi=0}^{\pi}\\ &= \left(\tfrac{3}{4} - \tfrac{1}{12}\right) - \left(-\tfrac{3}{4} + \tfrac{1}{12}\right) = \tfrac{2}{3} - \left(-\tfrac{2}{3}\right) = \tfrac{4}{3}.[2]\end{aligned}$$

Putting this together, we get

$$\iint_S \mathbf{v}.d\mathbf{A} = 8\times\tfrac{4}{3}\times\pi = \frac{32\pi}{3}.[1]$$

(c) First, we have

$$\iiint_E xyz\, dV = \int_{x=0}^{1}\int_{y=0}^{1}\int_{z=0}^{1-x} xyz\, dz\, dy\, dx[2]$$

The innermost integral is

$$\int_{z=0}^{1-x} xyz\, dz = \left[\frac{xyz^2}{2}\right]_{z=0}^{1-x} = \frac{x(1-x)^2y}{2},\ [1]$$

so the middle integral is thus

$$\int_{y=0}^{1}\frac{x(1-x)^2y}{2}\, dy = \left[\frac{x(1-x)^2y^2}{4}\right]_{y=0}^{1} = \frac{x(1-x)^2}{4} = \tfrac{1}{4}x - \tfrac{1}{2}x^2 + \tfrac{1}{4}x^3.[1]$$

For the outer integral this gives

$$\iiint_E xyz\, dV = \int_{x=0}^{1}\tfrac{1}{4}x - \tfrac{1}{2}x^2 + \tfrac{1}{4}x^3\, dx = \left[\tfrac{1}{8}x^2 - \tfrac{1}{6}x^3 + \tfrac{1}{16}x^4\right]_{x=0}^{1} = \frac{1}{48}.[1]$$