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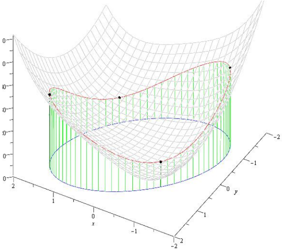
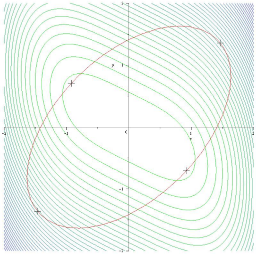
- The maximum and minimum occur at points where the road is tangent to the contours.





Constrained optimisation - applications

- Suppose we want to build a 5kW motor that is as light as possible. We have come up with a design with parameters a, b and c that we can adjust. The weight is $W(a, b, c)$ and the power (in kW) is $P(a, b, c)$. We want to minimise $W(a, b, c)$ subject to the constraint $P(a, b, c) - 5 = 0$.

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- More generally, whenever we design a device, there will be some requirements that are not negotiable; these will be expressed by constraint equations. There will be other functions that measure the effectiveness of the device. We want to maximise these, but we have to do so subject to the constraints.

The Lagrange multiplier method

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Geometric interpretation

$f(x, y) = x^2 + y^2 =$ squared distance from (x, y) to $(0, 0)$
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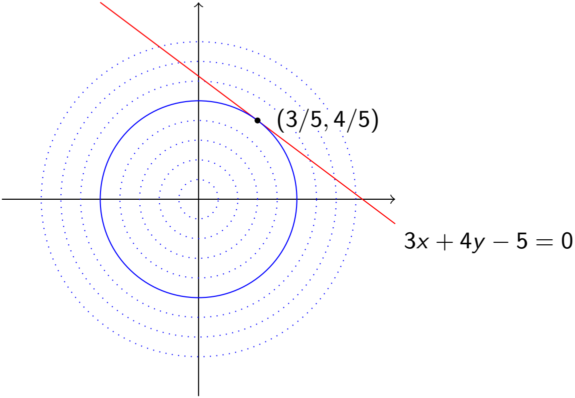
Geometrically, we have found the closest point to the origin on the line $3x + 4y = 5$.



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- Geometrically, the vector $\mathbf{u} = \begin{bmatrix} g_x \\ g_y \end{bmatrix}$ is normal to the constraint curve, and $\mathbf{v} = \begin{bmatrix} f_x \\ f_y \end{bmatrix}$ is normal to the contour of f. The equations $f_x = \lambda g_x$ and $f_y = \lambda g_y$ say that $\mathbf{v}$ is a multiple of $\mathbf{u}$, so the constraint curve is running parallel to the contour.

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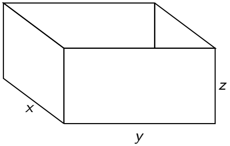
Consider a metal tank, open at the top. The volume is $V = xyz$, and the area is $S = xy + 2xz + 2yz$. We want the volume to be $4m^3$, and we want to minimise S, to use as little metal as possible.



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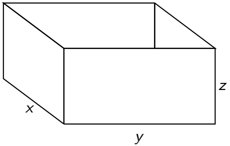

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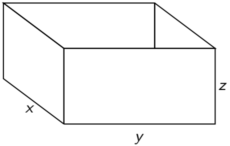

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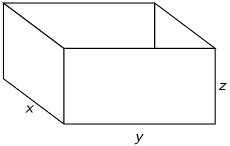

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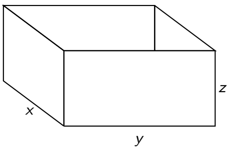

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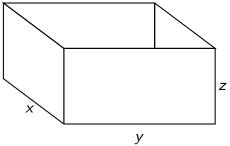

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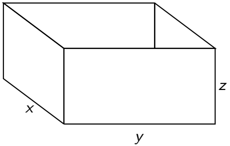

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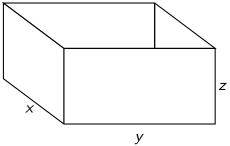

- We are minimising S subject to $V - 4 = 0$, so $L = xy + 2xz + 2yz - \lambda(xyz - 4)$.
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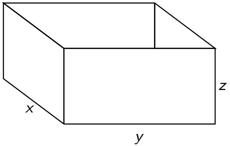

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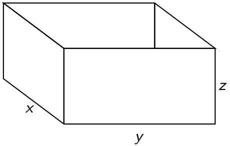

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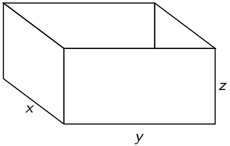

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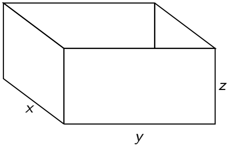

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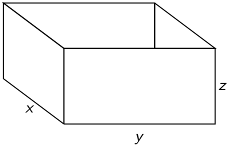

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- For these values, we have $S = 12$. Thus, the minimum possible area of metal sheet that we need is $12m^2$.

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This means that the maximum possible value of f (subject to the constraint) is $\sqrt{a+b}$, and the minimum is $-\sqrt{a+b}$.

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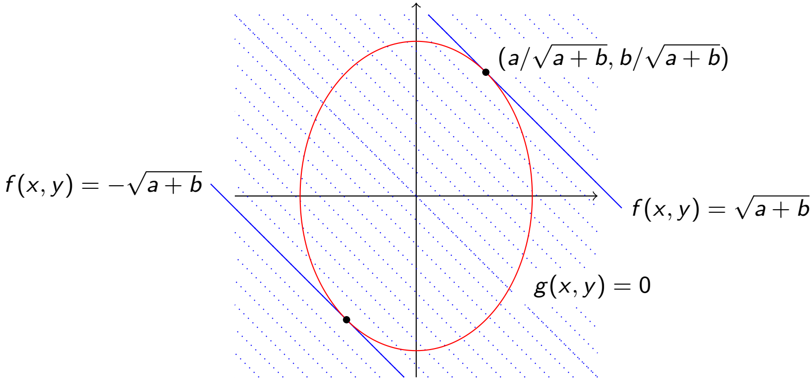
Problem: maximise $f(x, y) = x + y$ subject to $x^2/a + y^2/b = 1$
Maximum and minimum values are $\pm\sqrt{a+b}$, at the points $\pm(a, b)/\sqrt{a+b}$.



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- Thus, the critical points for the constrained problem are at $(-3, 0)$ (where $f = 0^2 - 8 \times (-3) + 17 = 41$) and $(3, 0)$ (where $f = -7$).

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- We conclude that the minimum is -7 and the maximum is 41.

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- For example: maximise z subject to $x^2 + y^2 + z^2 = 9$ and $x + 2y + 4z = 3$.

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$$(\lambda, \mu, x, y, z) = (1/12, 1/6, -1, -2, 2)$$
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- Thus, the minimum value of z is $-6/7$ at $(9/7, 18/7, -6/7)$, and the maximum is 2 at $(-1, 2, 2)$.

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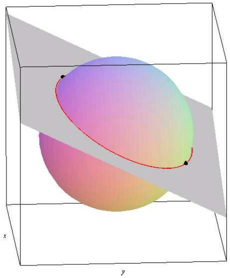

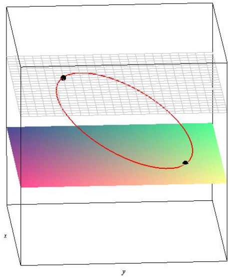

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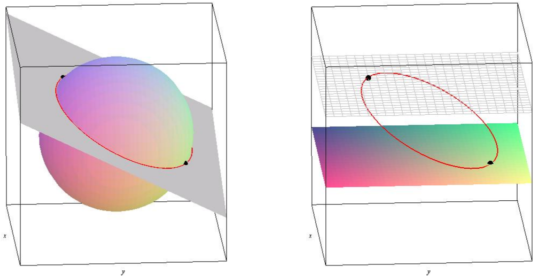

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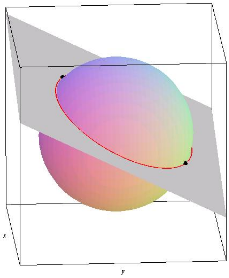

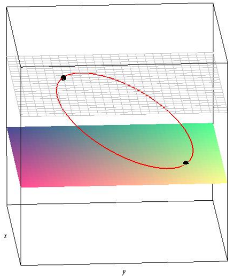

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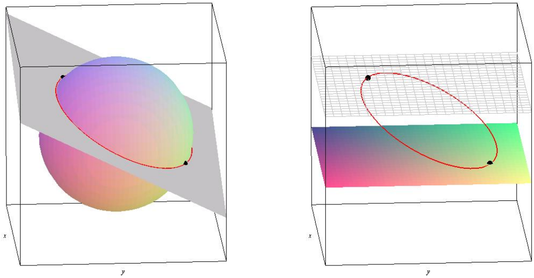

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