Integrals over plane regions

Let D be a region in the plane, and let $f(x, y)$ be a function defined for points (x, y) in D.

Integrals over plane regions

Let D be a region in the plane, and let $f(x, y)$ be a function defined for points (x, y) in D. We define the integral $\iint_D f(x, y)dA$ as follows.

Integrals over plane regions

Let D be a region in the plane, and let $f(x, y)$ be a function defined for points (x, y) in D. We define the integral $\iint_D f(x, y)dA$ as follows. First, we divide the region D into a large number of small regions $D_1, \ldots, D_N$.

Let D be a region in the plane, and let $f(x, y)$ be a function defined for points (x, y) in D. We define the integral $\iint_D f(x, y)dA$ as follows. First, we divide the region D into a large number of small regions $D_1, \ldots, D_N$. As each region D_i is small, the value of f will not change much as we move around D_i

Let D be a region in the plane, and let $f(x, y)$ be a function defined for points (x, y) in D. We define the integral $\iint_D f(x, y)dA$ as follows. First, we divide the region D into a large number of small regions $D_1, \ldots, D_N$. As each region D_i is small, the value of f will not change much as we move around D_i, so it makes approximate sense to talk about the value of f on D_i as a single number.

Let D be a region in the plane, and let $f(x, y)$ be a function defined for points (x, y) in D. We define the integral $\iint_D f(x, y) dA$ as follows. First, we divide the region D into a large number of small regions $D_1, \ldots, D_N$. As each region D_i is small, the value of f will not change much as we move around D_i, so it makes approximate sense to talk about the value of f on D_i as a single number. The integral is approximately defined by

$$\iint_D f(x, y)\, dA = \sum_{i=1}^{N} (\text{ value of } f \text{ on } D_i \text{ }) \times (\text{ area of } D_i).$$

Integrals over plane regions

Let D be a region in the plane, and let $f(x, y)$ be a function defined for points (x, y) in D. We define the integral $\iint_D f(x, y)dA$ as follows. First, we divide the region D into a large number of small regions $D_1, \ldots, D_N$. As each region D_i is small, the value of f will not change much as we move around D_i, so it makes approximate sense to talk about the value of f on D_i as a single number. The integral is approximately defined by

$$\iint_D f(x, y)\, dA = \sum_{i=1}^{N} (\text{ value of } f \text{ on } D_i\ ) \times (\text{ area of } D_i).$$

To get the exact value, we divide D into a larger and larger number of smaller and smaller pieces, and then pass to the limit.

Applications

Some applications of this kind of integration are as follows.

Applications

Some applications of this kind of integration are as follows.

(a) Suppose that the region D is a charged plate, and that the charge density at a point (x, y) is $q(x, y)$; then the total charge is $Q = \iint_D q(x, y)\, dA$.

Applications

Some applications of this kind of integration are as follows.

(a) Suppose that the region D is a charged plate, and that the charge density at a point (x, y) is $q(x, y)$; then the total charge is $Q = \iint_D q(x, y)\, dA$.

(b) Suppose that the region D represents a structure of constant density ρ and vertical thickness $f(x, y)$ attached to an axle passing vertically through the origin.

Applications

Some applications of this kind of integration are as follows.

(a) Suppose that the region D is a charged plate, and that the charge density at a point (x, y) is $q(x, y)$; then the total charge is $Q = \iint_D q(x, y)\, dA$.

(b) Suppose that the region D represents a structure of constant density ρ and vertical thickness $f(x, y)$ attached to an axle passing vertically through the origin. Then the mass of the structure is $\iint_D \rho f(x, y)\, dA$

Applications

Some applications of this kind of integration are as follows.

(a) Suppose that the region D is a charged plate, and that the charge density at a point (x, y) is $q(x, y)$; then the total charge is $Q = \iint_D q(x, y)\, dA$.

(b) Suppose that the region D represents a structure of constant density ρ and vertical thickness $f(x, y)$ attached to an axle passing vertically through the origin. Then the mass of the structure is $\iint_D \rho f(x, y)\, dA$, whereas the moment of inertia (which measures the difficulty of turning the axle) is $\iint_D \rho f(x, y)(x^2 + y^2) dA$.

Some applications of this kind of integration are as follows.

(a) Suppose that the region D is a charged plate, and that the charge density at a point (x, y) is $q(x, y)$; then the total charge is $Q = \iint_D q(x, y)\, dA$.

(b) Suppose that the region D represents a structure of constant density ρ and vertical thickness $f(x, y)$ attached to an axle passing vertically through the origin. Then the mass of the structure is $\iint_D \rho f(x, y)\, dA$, whereas the moment of inertia (which measures the difficulty of turning the axle) is $\iint_D \rho f(x, y)(x^2 + y^2) dA$.

(c) Suppose that the region D represents a large solar cell, with the brightness of light arriving at (x, y) being given by the function $f(x, y)$. Then the total incident power on the cell will be (a constant times) $\iint_D f(x, y)\, dA$.

Applications

Some applications of this kind of integration are as follows.

(a) Suppose that the region D is a charged plate, and that the charge density at a point (x, y) is $q(x, y)$; then the total charge is $Q = \iint_D q(x, y)\, dA$.

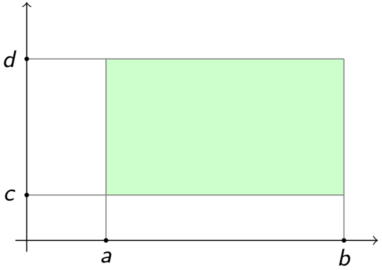
(b) Suppose that the region D represents a structure of constant density ρ and vertical thickness $f(x, y)$ attached to an axle passing vertically through the origin. Then the mass of the structure is $\iint_D \rho f(x, y)\, dA$, whereas the moment of inertia (which measures the difficulty of turning the axle) is $\iint_D \rho f(x, y)(x^2 + y^2) dA$.

(c) Suppose that the region D represents a large solar cell, with the brightness of light arriving at (x, y) being given by the function $f(x, y)$. Then the total incident power on the cell will be (a constant times) $\iint_D f(x, y)\, dA$.

(d) The total area of a region D is just $\iint_D 1\, dA$.

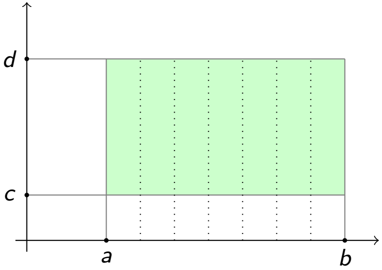
Rectangular regions

In the simplest case, the region D is a rectangle aligned with the axes, given by $a \leq x \leq b$ and $c \leq y \leq d$ say.



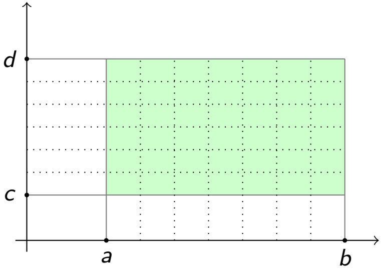
Rectangular regions

In the simplest case, the region D is a rectangle aligned with the axes, given by $a \leq x \leq b$ and $c \leq y \leq d$ say. In this case we can just divide the horizontal interval $[a, b]$ into small intervals of length δx



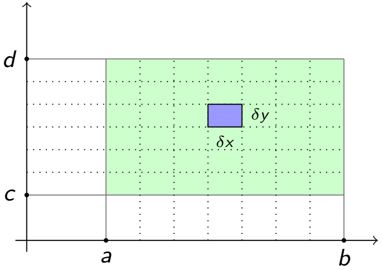
Rectangular regions

In the simplest case, the region D is a rectangle aligned with the axes, given by $a \leq x \leq b$ and $c \leq y \leq d$ say. In this case we can just divide the horizontal interval $[a, b]$ into small intervals of length δx, and divide the vertical interval $[c, d]$ into small intervals of length δy.



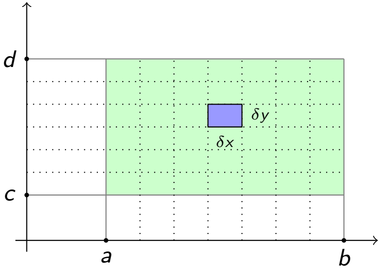
Rectangular regions

In the simplest case, the region D is a rectangle aligned with the axes, given by $a \leq x \leq b$ and $c \leq y \leq d$ say. In this case we can just divide the horizontal interval $[a, b]$ into small intervals of length δx, and divide the vertical interval $[c, d]$ into small intervals of length δy. This divides D into small rectangles of area $\delta A = \delta x . \delta y$.



Rectangular regions

In the simplest case, the region D is a rectangle aligned with the axes, given by $a \leq x \leq b$ and $c \leq y \leq d$ say. In this case we can just divide the horizontal interval $[a, b]$ into small intervals of length δx, and divide the vertical interval $[c, d]$ into small intervals of length δy. This divides D into small rectangles of area $\delta A = \delta x.\delta y$.

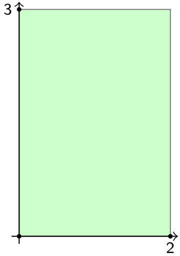


Using this kind of subdivision, we see that the area integral is just obtained by integrating with respect to both variables x and y:

$$\iint_D f(x, y)\, dA = \int_{x=a}^{b} \left(\int_{y=c}^{d} f(x, y)\, dy \right) dx.$$

Rectangular example

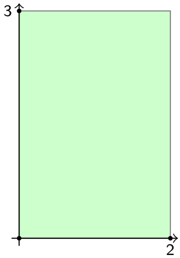
$D =$ rectangle where $0 \leq x \leq 2$ and $0 \leq y \leq 3$.



Rectangular example

$D =$ rectangle where $0 \leq x \leq 2$ and $0 \leq y \leq 3$.

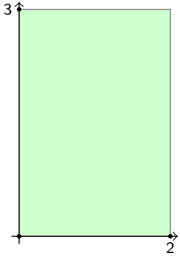
$$\iint_D x^3 + y^2 \, dA$$



Rectangular example

$D =$ rectangle where $0 \leq x \leq 2$ and $0 \leq y \leq 3$.

$$\iint_D x^3 + y^2\, dA = \int_{x=0}^{2} \left(\int_{y=0}^{3} x^3 + y^2\, dy \right) dx$$

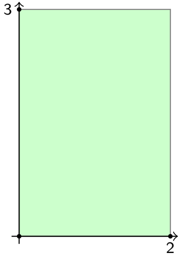


Rectangular example

$D =$ rectangle where $0 \leq x \leq 2$ and $0 \leq y \leq 3$.

$$\iint_D x^3 + y^2 \, dA = \int_{x=0}^{2} \left(\int_{y=0}^{3} x^3 + y^2 \, dy \right) dx$$

In the inner integral, we treat x as a constant and y as a variable.



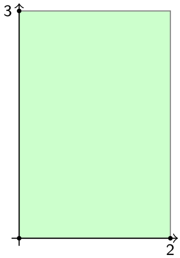
Rectangular example

$D =$ rectangle where $0 \leq x \leq 2$ and $0 \leq y \leq 3$.

$$\iint_D x^3 + y^2 \, dA = \int_{x=0}^{2} \left(\int_{y=0}^{3} x^3 + y^2 \, dy \right) dx$$

In the inner integral, we treat x as a constant and y as a variable. This gives

$$\int_{y=0}^{3} x^3 + y^2 \, dy = \left[x^3 y + y^3/3 \right]_{y=0}^{3} = 3x^3 + 9.$$



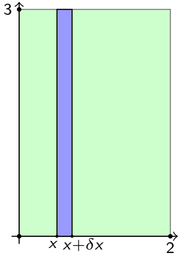
Rectangular example

$D =$ rectangle where $0 \leq x \leq 2$ and $0 \leq y \leq 3$.

$$\iint_D x^3 + y^2 \, dA = \int_{x=0}^{2} \left(\int_{y=0}^{3} x^3 + y^2 \, dy \right) dx$$

In the inner integral, we treat x as a constant and y as a variable. This gives

$$\int_{y=0}^{3} x^3 + y^2 \, dy = \left[x^3 y + y^3/3 \right]_{y=0}^{3} = 3x^3 + 9.$$



Meaning: if we take a thin strip running horizontally from x to $x + \delta x$, and vertically all the way from 0 to 3, then the sum of the corresponding contributions is approximately $(3x^3 + 9)\delta x$

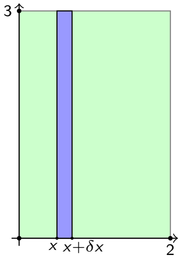
Rectangular example

$D =$ rectangle where $0 \leq x \leq 2$ and $0 \leq y \leq 3$.

$$\iint_D x^3 + y^2 \, dA = \int_{x=0}^{2} \left(\int_{y=0}^{3} x^3 + y^2 \, dy \right) dx$$

In the inner integral, we treat x as a constant and y as a variable. This gives

$$\int_{y=0}^{3} x^3 + y^2 \, dy = \left[x^3 y + y^3/3 \right]_{y=0}^{3} = 3x^3 + 9.$$



Meaning: if we take a thin strip running horizontally from x to $x + \delta x$, and vertically all the way from 0 to 3, then the sum of the corresponding contributions is approximately $(3x^3 + 9)\delta x$ (and the approximation becomes exact in the limit as $\delta x \to 0$).

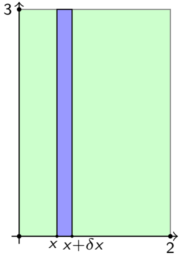
Rectangular example

$D =$ rectangle where $0 \leq x \leq 2$ and $0 \leq y \leq 3$.

$$\iint_D x^3 + y^2 \, dA = \int_{x=0}^{2} \left(\int_{y=0}^{3} x^3 + y^2 \, dy \right) dx$$

In the inner integral, we treat x as a constant and y as a variable. This gives

$$\int_{y=0}^{3} x^3 + y^2 \, dy = \left[x^3 y + y^3/3 \right]_{y=0}^{3} = 3x^3 + 9.$$



Meaning: if we take a thin strip running horizontally from x to $x + \delta x$, and vertically all the way from 0 to 3, then the sum of the corresponding contributions is approximately $(3x^3 + 9)\delta x$ (and the approximation becomes exact in the limit as $\delta x \to 0$). Outer integral: add up the contributions from all such vertical strips.

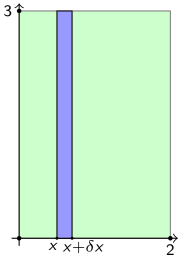
Rectangular example

$D =$ rectangle where $0 \leq x \leq 2$ and $0 \leq y \leq 3$.

$$\iint_D x^3 + y^2 \, dA = \int_{x=0}^{2} \left(\int_{y=0}^{3} x^3 + y^2 \, dy \right) dx$$

In the inner integral, we treat x as a constant and y as a variable. This gives

$$\int_{y=0}^{3} x^3 + y^2 \, dy = \left[x^3 y + y^3/3 \right]_{y=0}^{3} = 3x^3 + 9.$$



Meaning: if we take a thin strip running horizontally from x to $x + \delta x$, and vertically all the way from 0 to 3, then the sum of the corresponding contributions is approximately $(3x^3 + 9)\delta x$ (and the approximation becomes exact in the limit as $\delta x \to 0$). Outer integral: add up the contributions from all such vertical strips.

$$\int_{x=0}^{2} 3x^3 + 9 \, dx$$

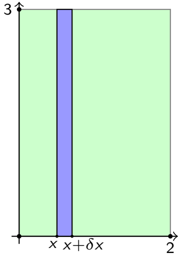
Rectangular example

$D =$ rectangle where $0 \le x \le 2$ and $0 \le y \le 3$.

$$\iint_D x^3 + y^2 \, dA = \int_{x=0}^{2} \left(\int_{y=0}^{3} x^3 + y^2 \, dy \right) dx$$

In the inner integral, we treat x as a constant and y as a variable. This gives

$$\int_{y=0}^{3} x^3 + y^2 \, dy = \left[x^3 y + y^3/3 \right]_{y=0}^{3} = 3x^3 + 9.$$



Meaning: if we take a thin strip running horizontally from x to $x + \delta x$, and vertically all the way from 0 to 3, then the sum of the corresponding contributions is approximately $(3x^3 + 9)\delta x$ (and the approximation becomes exact in the limit as $\delta x \to 0$). Outer integral: add up the contributions from all such vertical strips.

$$\int_{x=0}^{2} 3x^3 + 9 \, dx = \left[3x^4/4 + 9x \right]_{x=0}^{2}$$

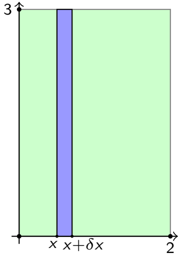
Rectangular example

D = rectangle where $0 \leq x \leq 2$ and $0 \leq y \leq 3$.

$$\iint_D x^3 + y^2 \, dA = \int_{x=0}^{2} \left(\int_{y=0}^{3} x^3 + y^2 \, dy \right) dx$$

In the inner integral, we treat x as a constant and y as a variable. This gives

$$\int_{y=0}^{3} x^3 + y^2 \, dy = \left[x^3 y + y^3/3 \right]_{y=0}^{3} = 3x^3 + 9.$$



Meaning: if we take a thin strip running horizontally from x to $x + \delta x$, and vertically all the way from 0 to 3, then the sum of the corresponding contributions is approximately $(3x^3 + 9)\delta x$ (and the approximation becomes exact in the limit as $\delta x \to 0$). Outer integral: add up the contributions from all such vertical strips.

$$\int_{x=0}^{2} 3x^3 + 9 \, dx = \left[3x^4/4 + 9x \right]_{x=0}^{2} = (12 + 18) - (0) = 30.$$

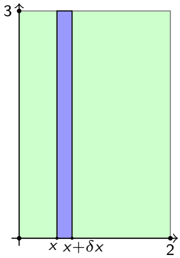
Rectangular example

$D =$ rectangle where $0 \le x \le 2$ and $0 \le y \le 3$.

$$\iint_D x^3 + y^2\,dA = \int_{x=0}^{2} \left(\int_{y=0}^{3} x^3 + y^2\,dy \right) dx$$

In the inner integral, we treat x as a constant and y as a variable. This gives

$$\int_{y=0}^{3} x^3 + y^2\,dy = \left[x^3 y + y^3/3 \right]_{y=0}^{3} = 3x^3 + 9.$$



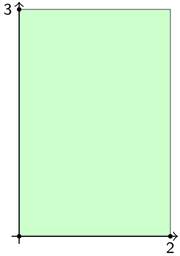
Meaning: if we take a thin strip running horizontally from x to $x + \delta x$, and vertically all the way from 0 to 3, then the sum of the corresponding contributions is approximately $(3x^3 + 9)\delta x$ (and the approximation becomes exact in the limit as $\delta x \to 0$). Outer integral: add up the contributions from all such vertical strips.

$$\int_{x=0}^{2} 3x^3 + 9\,dx = \left[3x^4/4 + 9x \right]_{x=0}^{2} = (12 + 18) - (0) = 30.$$

The conclusion is that $\iint_D x^3 + y^2\,dA = 30$.

Rectangular example — horizontal strips

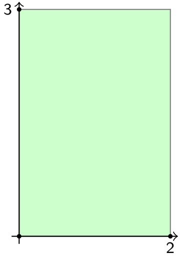
$D =$ rectangle where $0 \leq x \leq 2$ and $0 \leq y \leq 3$.



Rectangular example — horizontal strips

D = rectangle where $0 \le x \le 2$ and $0 \le y \le 3$.

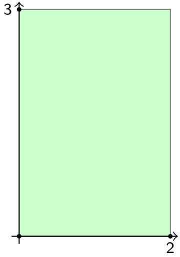
$$\iint_D x^3 + y^2 \, dA$$



Rectangular example — horizontal strips

$D =$ rectangle where $0 \leq x \leq 2$ and $0 \leq y \leq 3$.

$$\iint_D x^3 + y^2 \, dA = \int_{y=0}^{3} \left(\int_{x=0}^{2} x^3 + y^2 \, dx \right) dy$$

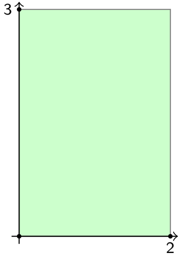


Rectangular example — horizontal strips

$D =$ rectangle where $0 \leq x \leq 2$ and $0 \leq y \leq 3$.

$$\iint_D x^3 + y^2 \, dA = \int_{y=0}^{3} \left(\int_{x=0}^{2} x^3 + y^2 \, dx \right) dy$$

In the inner integral, we treat y as a constant and x as a variable.



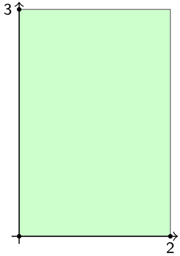
Rectangular example — horizontal strips

$D =$ rectangle where $0 \leq x \leq 2$ and $0 \leq y \leq 3$.

$$\iint_D x^3 + y^2\, dA = \int_{y=0}^{3} \left(\int_{x=0}^{2} x^3 + y^2\, dx \right) dy$$

In the inner integral, we treat y as a constant and x as a variable. This gives

$$\int_{x=0}^{2} x^3 + y^2\, dx = \left[x^4/4 + xy^2 \right]_{x=0}^{2} = 4 + 2y^2.$$



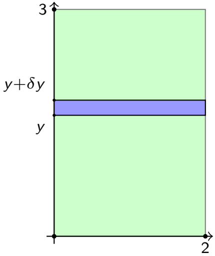
Rectangular example — horizontal strips

$D =$ rectangle where $0 \leq x \leq 2$ and $0 \leq y \leq 3$.

$$\iint_D x^3 + y^2 \, dA = \int_{y=0}^{3} \left(\int_{x=0}^{2} x^3 + y^2 \, dx \right) dy$$

In the inner integral, we treat y as a constant and x as a variable. This gives

$$\int_{x=0}^{2} x^3 + y^2 \, dx = \left[x^4/4 + xy^2 \right]_{x=0}^{2} = 4 + 2y^2.$$



Meaning: if we take a thin strip running vertically from y to $y + \delta y$, and horizontally all the way from 0 to 2, then the sum of the corresponding contributions is approximately $(4 + 2y^2)\delta y$

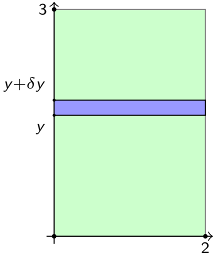
Rectangular example — horizontal strips

$D =$ rectangle where $0 \leq x \leq 2$ and $0 \leq y \leq 3$.

$$\iint_D x^3 + y^2 \, dA = \int_{y=0}^{3} \left(\int_{x=0}^{2} x^3 + y^2 \, dx \right) dy$$

In the inner integral, we treat y as a constant and x as a variable. This gives

$$\int_{x=0}^{2} x^3 + y^2 \, dx = \left[x^4/4 + xy^2 \right]_{x=0}^{2} = 4 + 2y^2.$$



Meaning: if we take a thin strip running vertically from y to $y + \delta y$, and horizontally all the way from 0 to 2, then the sum of the corresponding contributions is approximately $(4 + 2y^2)\delta y$ (and the approximation becomes exact in the limit as $\delta y \to 0$).

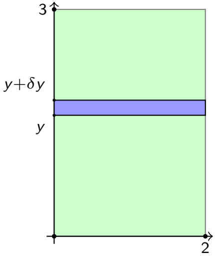
Rectangular example — horizontal strips

$D =$ rectangle where $0 \leq x \leq 2$ and $0 \leq y \leq 3$.

$$\iint_D x^3 + y^2 \, dA = \int_{y=0}^{3} \left(\int_{x=0}^{2} x^3 + y^2 \, dx \right) dy$$

In the inner integral, we treat y as a constant and x as a variable. This gives

$$\int_{x=0}^{2} x^3 + y^2 \, dx = \left[x^4/4 + xy^2 \right]_{x=0}^{2} = 4 + 2y^2.$$



Meaning: if we take a thin strip running vertically from y to $y + \delta y$, and horizontally all the way from 0 to 2, then the sum of the corresponding contributions is approximately $(4 + 2y^2)\delta y$ (and the approximation becomes exact in the limit as $\delta y \to 0$). Outer integral: add up the contributions from all such horizontal strips.

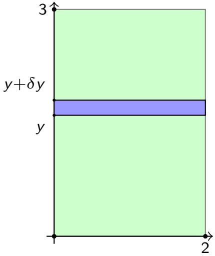
Rectangular example — horizontal strips

$D =$ rectangle where $0 \leq x \leq 2$ and $0 \leq y \leq 3$.

$$\iint_D x^3 + y^2 \, dA = \int_{y=0}^{3} \left(\int_{x=0}^{2} x^3 + y^2 \, dx \right) dy$$

In the inner integral, we treat y as a constant and x as a variable. This gives

$$\int_{x=0}^{2} x^3 + y^2 \, dx = \left[x^4/4 + xy^2 \right]_{x=0}^{2} = 4 + 2y^2.$$



Meaning: if we take a thin strip running vertically from y to $y + \delta y$, and horizontally all the way from 0 to 2, then the sum of the corresponding contributions is approximately $(4 + 2y^2)\delta y$ (and the approximation becomes exact in the limit as $\delta y \to 0$). Outer integral: add up the contributions from all such horizontal strips.

$$\int_{y=0}^{3} 4 + 2y^2 \, dy$$

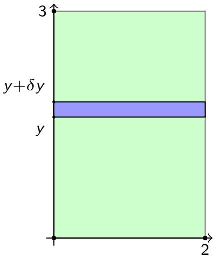
Rectangular example — horizontal strips

D = rectangle where $0 \leq x \leq 2$ and $0 \leq y \leq 3$.

$$\iint_D x^3 + y^2 \, dA = \int_{y=0}^{3} \left(\int_{x=0}^{2} x^3 + y^2 \, dx \right) dy$$

In the inner integral, we treat y as a constant and x as a variable. This gives

$$\int_{x=0}^{2} x^3 + y^2 \, dx = \left[x^4/4 + xy^2 \right]_{x=0}^{2} = 4 + 2y^2.$$



Meaning: if we take a thin strip running vertically from y to $y + \delta y$, and horizontally all the way from 0 to 2, then the sum of the corresponding contributions is approximately $(4 + 2y^2)\delta y$ (and the approximation becomes exact in the limit as $\delta y \to 0$). Outer integral: add up the contributions from all such horizontal strips.

$$\int_{y=0}^{3} 4 + 2y^2 \, dy = \left[4y + 2y^3/3 \right]_{y=0}^{3}$$

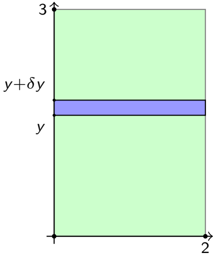
Rectangular example — horizontal strips

$D =$ rectangle where $0 \leq x \leq 2$ and $0 \leq y \leq 3$.

$$\iint_D x^3 + y^2 \, dA = \int_{y=0}^{3} \left(\int_{x=0}^{2} x^3 + y^2 \, dx \right) dy$$

In the inner integral, we treat y as a constant and x as a variable. This gives

$$\int_{x=0}^{2} x^3 + y^2 \, dx = \left[x^4/4 + xy^2 \right]_{x=0}^{2} = 4 + 2y^2.$$



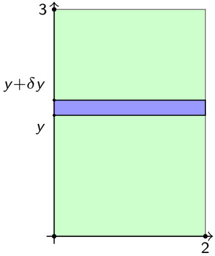
Meaning: if we take a thin strip running vertically from y to $y + \delta y$, and horizontally all the way from 0 to 2, then the sum of the corresponding contributions is approximately $(4 + 2y^2)\delta y$ (and the approximation becomes exact in the limit as $\delta y \to 0$). Outer integral: add up the contributions from all such horizontal strips.

$$\int_{y=0}^{3} 4 + 2y^2 \, dy = \left[4y + 2y^3/3 \right]_{y=0}^{3} = (12 + 2 \times 27/3) - (0) = 30.$$

Rectangular example — horizontal strips

$D =$ rectangle where $0 \leq x \leq 2$ and $0 \leq y \leq 3$.

$$\iint_D x^3 + y^2 \, dA = \int_{y=0}^{3} \left(\int_{x=0}^{2} x^3 + y^2 \, dx \right) dy$$



In the inner integral, we treat y as a constant and x as a variable. This gives

$$\int_{x=0}^{2} x^3 + y^2 \, dx = \left[x^4/4 + xy^2 \right]_{x=0}^{2} = 4 + 2y^2.$$

Meaning: if we take a thin strip running vertically from y to $y + \delta y$, and horizontally all the way from 0 to 2, then the sum of the corresponding contributions is approximately $(4 + 2y^2)\delta y$ (and the approximation becomes exact in the limit as $\delta y \to 0$). Outer integral: add up the contributions from all such horizontal strips.

$$\int_{y=0}^{3} 4 + 2y^2 \, dy = \left[4y + 2y^3/3 \right]_{y=0}^{3} = (12 + 2 \times 27/3) - (0) = 30.$$

The conclusion is again that $\iint_D x^3 + y^2 \, dA = 30$.

Square region example

Let E be the square where $0 \leq x \leq \pi$ and $-\pi/2 \leq y \leq \pi/2$.

Square region example

Let E be the square where $0 \leq x \leq \pi$ and $-\pi/2 \leq y \leq \pi/2$.

$$\iint_E \sin(x)\cos(y)\, dA$$

Square region example

Let E be the square where $0 \leq x \leq \pi$ and $-\pi/2 \leq y \leq \pi/2$.

$$\iint_E \sin(x)\cos(y)\,dA = \int_{x=0}^{\pi} \left(\int_{y=-\pi/2}^{\pi/2} \sin(x)\cos(y)\,dy \right) dx.$$

Square region example

Let E be the square where $0 \leq x \leq \pi$ and $-\pi/2 \leq y \leq \pi/2$.

$$\iint_E \sin(x)\cos(y)\,dA = \int_{x=0}^{\pi} \left(\int_{y=-\pi/2}^{\pi/2} \sin(x)\cos(y)\,dy \right) dx.$$

In the inner integral, we treat x as a constant and y as a variable.

Square region example

Let E be the square where $0 \leq x \leq \pi$ and $-\pi/2 \leq y \leq \pi/2$.

$$\iint_E \sin(x)\cos(y)\,dA = \int_{x=0}^{\pi} \left(\int_{y=-\pi/2}^{\pi/2} \sin(x)\cos(y)\,dy \right) dx.$$

In the inner integral, we treat x as a constant and y as a variable. This gives

$$\int_{y=-\pi/2}^{\pi/2} \sin(x)\cos(y)\,dy = \sin(x)\,[\sin(y)]_{y=-\pi/2}^{\pi/2}$$

Square region example

Let E be the square where $0 \leq x \leq \pi$ and $-\pi/2 \leq y \leq \pi/2$.

$$\iint_E \sin(x)\cos(y)\,dA = \int_{x=0}^{\pi} \left(\int_{y=-\pi/2}^{\pi/2} \sin(x)\cos(y)\,dy \right) dx.$$

In the inner integral, we treat x as a constant and y as a variable. This gives

$$\int_{y=-\pi/2}^{\pi/2} \sin(x)\cos(y)\,dy = \sin(x)\,[\sin(y)]_{y=-\pi/2}^{\pi/2} = \sin(x)(1-(-1)) = 2\sin(x).$$

Square region example

Let E be the square where $0 \leq x \leq \pi$ and $-\pi/2 \leq y \leq \pi/2$.

$$\iint_E \sin(x)\cos(y)\,dA = \int_{x=0}^{\pi} \left(\int_{y=-\pi/2}^{\pi/2} \sin(x)\cos(y)\,dy \right) dx.$$

In the inner integral, we treat x as a constant and y as a variable. This gives

$$\int_{y=-\pi/2}^{\pi/2} \sin(x)\cos(y)\,dy = \sin(x)\,[\sin(y)]_{y=-\pi/2}^{\pi/2} = \sin(x)(1-(-1)) = 2\sin(x).$$

Again, this means that the contribution coming from a vertical strip of width δx is approximately $2\sin(x)\delta x$.

Square region example

Let E be the square where $0 \leq x \leq \pi$ and $-\pi/2 \leq y \leq \pi/2$.

$$\iint_E \sin(x)\cos(y)\, dA = \int_{x=0}^{\pi} \left(\int_{y=-\pi/2}^{\pi/2} \sin(x)\cos(y)\, dy \right) dx.$$

In the inner integral, we treat x as a constant and y as a variable. This gives

$$\int_{y=-\pi/2}^{\pi/2} \sin(x)\cos(y)\, dy = \sin(x)\left[\sin(y)\right]_{y=-\pi/2}^{\pi/2} = \sin(x)(1-(-1)) = 2\sin(x).$$

Again, this means that the contribution coming from a vertical strip of width δx is approximately $2\sin(x)\delta x$. We can now perform the outer integral to add up the contributions from all such vertical strips:

$$\int_{x=0}^{\pi} 2\sin(x)\, dx$$

Square region example

Let E be the square where $0 \leq x \leq \pi$ and $-\pi/2 \leq y \leq \pi/2$.

$$\iint_E \sin(x)\cos(y)\,dA = \int_{x=0}^{\pi} \left(\int_{y=-\pi/2}^{\pi/2} \sin(x)\cos(y)\,dy \right) dx.$$

In the inner integral, we treat x as a constant and y as a variable. This gives

$$\int_{y=-\pi/2}^{\pi/2} \sin(x)\cos(y)\,dy = \sin(x)\,[\sin(y)]_{y=-\pi/2}^{\pi/2} = \sin(x)(1-(-1)) = 2\sin(x).$$

Again, this means that the contribution coming from a vertical strip of width δx is approximately $2\sin(x)\delta x$. We can now perform the outer integral to add up the contributions from all such vertical strips:

$$\int_{x=0}^{\pi} 2\sin(x)\,dx = 2\,[-\cos(x)]_0^{\pi}$$

Square region example

Let E be the square where $0 \leq x \leq \pi$ and $-\pi/2 \leq y \leq \pi/2$.

$$\iint_E \sin(x)\cos(y)\, dA = \int_{x=0}^{\pi} \left(\int_{y=-\pi/2}^{\pi/2} \sin(x)\cos(y)\, dy \right) dx.$$

In the inner integral, we treat x as a constant and y as a variable. This gives

$$\int_{y=-\pi/2}^{\pi/2} \sin(x)\cos(y)\, dy = \sin(x)\left[\sin(y)\right]_{y=-\pi/2}^{\pi/2} = \sin(x)(1-(-1)) = 2\sin(x).$$

Again, this means that the contribution coming from a vertical strip of width δx is approximately $2\sin(x)\delta x$. We can now perform the outer integral to add up the contributions from all such vertical strips:

$$\int_{x=0}^{\pi} 2\sin(x)\, dx = 2\left[-\cos(x)\right]_0^{\pi} = 2(1-(-1)) = 4.$$

Square region example

Let E be the square where $0 \leq x \leq \pi$ and $-\pi/2 \leq y \leq \pi/2$.

$$\iint_E \sin(x)\cos(y)\,dA = \int_{x=0}^{\pi}\left(\int_{y=-\pi/2}^{\pi/2} \sin(x)\cos(y)\,dy\right)dx.$$

In the inner integral, we treat x as a constant and y as a variable. This gives

$$\int_{y=-\pi/2}^{\pi/2} \sin(x)\cos(y)\,dy = \sin(x)\left[\sin(y)\right]_{y=-\pi/2}^{\pi/2} = \sin(x)(1-(-1)) = 2\sin(x).$$

Again, this means that the contribution coming from a vertical strip of width δx is approximately $2\sin(x)\delta x$. We can now perform the outer integral to add up the contributions from all such vertical strips:

$$\int_{x=0}^{\pi} 2\sin(x)\,dx = 2\left[-\cos(x)\right]_0^{\pi} = 2(1-(-1)) = 4.$$

The conclusion is that $\iint_E \sin(x)\cos(y)\,dA = 4$.

Answer is constant

$$\iint_D x^3 + y^2 \, dA = 30$$

$$\iint_E \sin(x)\cos(y) \, dA = 4.$$

Note that in the last two examples, the final answer is just a number, not a function of x and y.

Answer is constant

$$\iint_D x^3 + y^2 \, dA = 30$$

$$\iint_E \sin(x)\cos(y) \, dA = 4.$$

Note that in the last two examples, the final answer is just a number, not a function of x and y. We have integrated over all relevant values of x and y, so there is no remaining dependence on x and y.

Answer is constant

$$\iint_D x^3 + y^2 \, dA = 30$$

$$\iint_E \sin(x)\cos(y) \, dA = 4.$$

Note that in the last two examples, the final answer is just a number, not a function of x and y. We have integrated over all relevant values of x and y, so there is no remaining dependence on x and y. Some common mistakes lead to an answer that does depend on x and/or y

Answer is constant

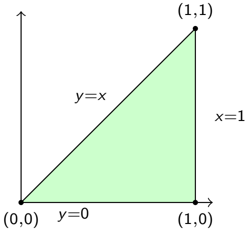
$$\iint_D x^3 + y^2 \, dA = 30$$

$$\iint_E \sin(x)\cos(y) \, dA = 4.$$

Note that in the last two examples, the final answer is just a number, not a function of x and y. We have integrated over all relevant values of x and y, so there is no remaining dependence on x and y. Some common mistakes lead to an answer that does depend on x and/or y; if you get such an answer, you need to look for the mistake.

Triangular example

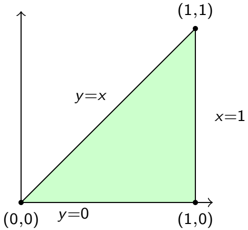
$D =$ triangle with vertices $(0,0)$, $(1,0)$ and $(1,1)$.



Triangular example

D = triangle with vertices $(0,0)$, $(1,0)$ and $(1,1)$.

$$\iint_D e^{2x-2y}\, dA$$

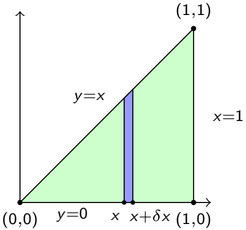


Triangular example

$D =$ triangle with vertices $(0,0)$, $(1,0)$ and $(1,1)$.

$$\iint_D e^{2x-2y}\, dA = \int_{x=0}^{1} \left(\int_{y=0}^{x} e^{2x-2y}\, dy \right) dx$$

Limits in the inner integral are the range of y values for a particular x.

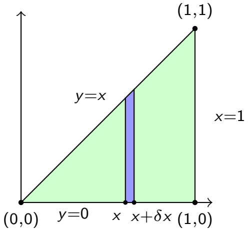


Triangular example

$D =$ triangle with vertices $(0,0)$, $(1,0)$ and $(1,1)$.

$$\iint_D e^{2x-2y}\, dA = \int_{x=0}^{1} \left(\int_{y=0}^{x} e^{2x-2y}\, dy \right) dx$$

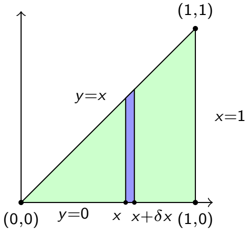
Limits in the inner integral are the range of y values for a particular x. In this integral, we treat x as a constant and y as a variable.



Triangular example

$D =$ triangle with vertices $(0,0)$, $(1,0)$ and $(1,1)$.

$$\iint_D e^{2x-2y}\,dA = \int_{x=0}^{1}\left(\int_{y=0}^{x} e^{2x-2y}\,dy\right)dx$$



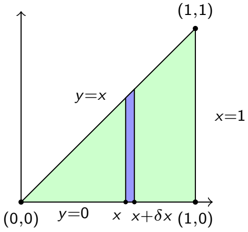
Limits in the inner integral are the range of y values for a particular x. In this integral, we treat x as a constant and y as a variable. This gives

$$\int_{y=0}^{x} e^{2x-2y}\,dy = \left[e^{2x-2y}/(-2)\right]_{y=0}^{x} = (e^{0}-e^{2x})/(-2)$$

Triangular example

D = triangle with vertices $(0,0)$, $(1,0)$ and $(1,1)$.

$$\iint_D e^{2x-2y}\,dA = \int_{x=0}^{1}\left(\int_{y=0}^{x} e^{2x-2y}\,dy\right)dx$$



Limits in the inner integral are the range of y values for a particular x. In this integral, we treat x as a constant and y as a variable. This gives

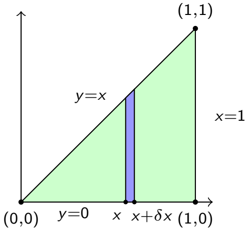
$$\int_{y=0}^{x} e^{2x-2y}\,dy = \left[e^{2x-2y}/(-2)\right]_{y=0}^{x} = (e^{0}-e^{2x})/(-2)$$

$$= \tfrac{1}{2}(e^{2x}-1).$$

Triangular example

$D =$ triangle with vertices $(0,0)$, $(1,0)$ and $(1,1)$.

$$\iint_D e^{2x-2y}\,dA = \int_{x=0}^{1}\left(\int_{y=0}^{x} e^{2x-2y}\,dy\right)dx$$



Limits in the inner integral are the range of y values for a particular x. In this integral, we treat x as a constant and y as a variable. This gives

$$\int_{y=0}^{x} e^{2x-2y}\,dy = \left[e^{2x-2y}/(-2)\right]_{y=0}^{x} = (e^0 - e^{2x})/(-2)$$

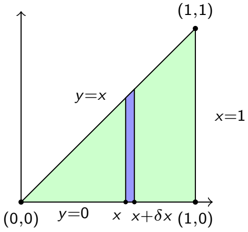
$$= \tfrac{1}{2}(e^{2x}-1).$$

Limits for the outer integral are the full range of x values anywhere in the region

Triangular example

$D =$ triangle with vertices $(0,0)$, $(1,0)$ and $(1,1)$.

$$\iint_D e^{2x-2y}\,dA = \int_{x=0}^{1}\left(\int_{y=0}^{x} e^{2x-2y}\,dy\right)dx$$



Limits in the inner integral are the range of y values for a particular x. In this integral, we treat x as a constant and y as a variable. This gives

$$\int_{y=0}^{x} e^{2x-2y}\,dy = \left[e^{2x-2y}/(-2)\right]_{y=0}^{x} = (e^0 - e^{2x})/(-2)$$

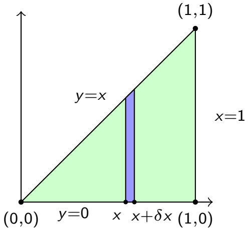
$$= \tfrac{1}{2}(e^{2x} - 1).$$

Limits for the outer integral are the full range of x values anywhere in the region, which means $0 \le x \le 1$ in this example.

Triangular example

$D =$ triangle with vertices $(0,0)$, $(1,0)$ and $(1,1)$.

$$\iint_D e^{2x-2y}\,dA = \int_{x=0}^{1}\left(\int_{y=0}^{x} e^{2x-2y}\,dy\right)dx$$



Limits in the inner integral are the range of y values for a particular x. In this integral, we treat x as a constant and y as a variable. This gives

$$\int_{y=0}^{x} e^{2x-2y}\,dy = \left[e^{2x-2y}/(-2)\right]_{y=0}^{x} = (e^{0}-e^{2x})/(-2)$$

$= \frac{1}{2}(e^{2x}-1)$.

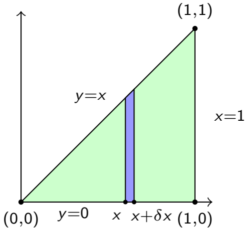
Limits for the outer integral are the full range of x values anywhere in the region, which means $0 \le x \le 1$ in this example.

$$\iint_D e^{2x-2y}\,dA = \int_{x=0}^{1} \tfrac{1}{2}(e^{2x}-1)\,dx$$

Triangular example

D = triangle with vertices $(0,0)$, $(1,0)$ and $(1,1)$.

$$\iint_D e^{2x-2y}\,dA = \int_{x=0}^{1}\left(\int_{y=0}^{x} e^{2x-2y}\,dy\right)dx$$



Limits in the inner integral are the range of y values for a particular x. In this integral, we treat x as a constant and y as a variable. This gives

$$\int_{y=0}^{x} e^{2x-2y}\,dy = \left[e^{2x-2y}/(-2)\right]_{y=0}^{x} = (e^0 - e^{2x})/(-2)$$

$$= \tfrac{1}{2}(e^{2x}-1).$$

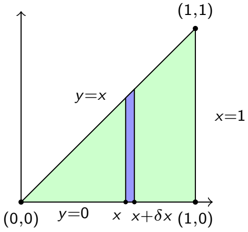
Limits for the outer integral are the full range of x values anywhere in the region, which means $0 \le x \le 1$ in this example.

$$\iint_D e^{2x-2y}\,dA = \int_{x=0}^{1} \tfrac{1}{2}(e^{2x}-1)\,dx = \left[\tfrac{1}{2}(\tfrac{1}{2}e^{2x} - x)\right]_{x=0}^{1}$$

Triangular example

D = triangle with vertices $(0,0)$, $(1,0)$ and $(1,1)$.

$$\iint_D e^{2x-2y}\,dA = \int_{x=0}^{1}\left(\int_{y=0}^{x} e^{2x-2y}\,dy\right)dx$$



Limits in the inner integral are the range of y values for a particular x. In this integral, we treat x as a constant and y as a variable. This gives

$$\int_{y=0}^{x} e^{2x-2y}\,dy = \left[e^{2x-2y}/(-2)\right]_{y=0}^{x} = (e^{0}-e^{2x})/(-2)$$

$$= \tfrac{1}{2}(e^{2x}-1).$$

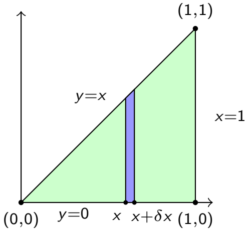
Limits for the outer integral are the full range of x values anywhere in the region, which means $0 \le x \le 1$ in this example.

$$\iint_D e^{2x-2y}\,dA = \int_{x=0}^{1} \tfrac{1}{2}(e^{2x}-1)\,dx = \left[\tfrac{1}{2}\left(\tfrac{1}{2}e^{2x}-x\right)\right]_{x=0}^{1} = \tfrac{1}{2}\left(\tfrac{1}{2}e^{2}-1\right)-\tfrac{1}{2}\left(\tfrac{1}{2}-0\right)$$

Triangular example

D = triangle with vertices $(0,0)$, $(1,0)$ and $(1,1)$.

$$\iint_D e^{2x-2y}\,dA = \int_{x=0}^{1}\left(\int_{y=0}^{x} e^{2x-2y}\,dy\right)dx$$



Limits in the inner integral are the range of y values for a particular x. In this integral, we treat x as a constant and y as a variable. This gives

$$\int_{y=0}^{x} e^{2x-2y}\,dy = \left[e^{2x-2y}/(-2)\right]_{y=0}^{x} = (e^0 - e^{2x})/(-2)$$

$$= \tfrac{1}{2}(e^{2x}-1).$$

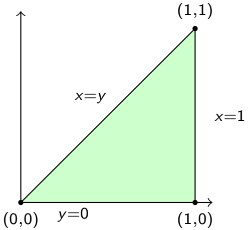
Limits for the outer integral are the full range of x values anywhere in the region, which means $0 \leq x \leq 1$ in this example.

$$\iint_D e^{2x-2y}\,dA = \int_{x=0}^{1} \tfrac{1}{2}(e^{2x}-1)\,dx = \left[\tfrac{1}{2}(\tfrac{1}{2}e^{2x}-x)\right]_{x=0}^{1} = \tfrac{1}{2}(\tfrac{1}{2}e^2-1)-\tfrac{1}{2}(\tfrac{1}{2}-0)$$

$$= (e^2-3)/4.$$

Triangular example — horizontal strips

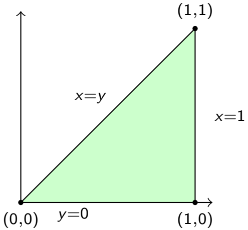
$D =$ triangle with vertices $(0,0)$, $(1,0)$ and $(1,1)$.



Triangular example — horizontal strips

$D =$ triangle with vertices $(0,0)$, $(1,0)$ and $(1,1)$.

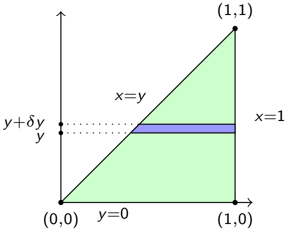
$$\iint_D e^{2x-2y}\, dA$$



Triangular example — horizontal strips

$D =$ triangle with vertices $(0,0)$, $(1,0)$ and $(1,1)$.

$$\iint_D e^{2x-2y}\,dA = \int_{y=0}^{1}\left(\int_{x=y}^{1} e^{2x-2y}\,dx\right)dy$$



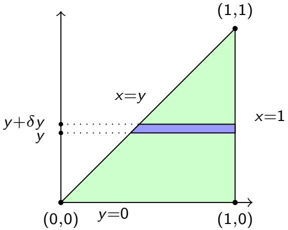
Limits in the inner integral are the range of x values for a particular y.

Triangular example — horizontal strips

$D =$ triangle with vertices $(0,0)$, $(1,0)$ and $(1,1)$.

$$\iint_D e^{2x-2y}\,dA = \int_{y=0}^{1}\left(\int_{x=y}^{1} e^{2x-2y}\,dx\right)dy$$

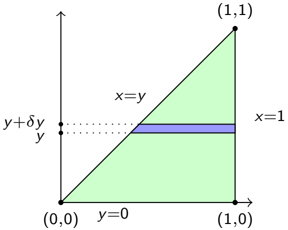
Limits in the inner integral are the range of x values for a particular y. In this integral, we treat y as a constant and x as a variable.



Triangular example — horizontal strips

D = triangle with vertices $(0, 0)$, $(1, 0)$ and $(1, 1)$.

$$\iint_D e^{2x-2y}\, dA = \int_{y=0}^{1} \left(\int_{x=y}^{1} e^{2x-2y}\, dx \right) dy$$



Limits in the inner integral are the range of x values for a particular y. In this integral, we treat y as a constant and x as a variable. This gives

$$\int_{x=y}^{1} e^{2x-2y}\, dx = \left[\tfrac{1}{2} e^{2x-2y} \right]_{x=y}^{1} = \tfrac{1}{2}(e^{2-2y} - 1)$$

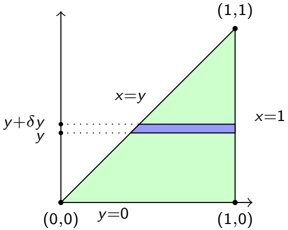
Triangular example — horizontal strips

$D =$ triangle with vertices $(0,0)$, $(1,0)$ and $(1,1)$.

$$\iint_D e^{2x-2y}\,dA = \int_{y=0}^{1}\left(\int_{x=y}^{1} e^{2x-2y}\,dx\right)dy$$

Limits in the inner integral are the range of x values for a particular y. In this integral, we treat y as a constant and x as a variable. This gives

$$\int_{x=y}^{1} e^{2x-2y}\,dx = \left[\tfrac{1}{2}e^{2x-2y}\right]_{x=y}^{1} = \tfrac{1}{2}(e^{2-2y}-1)$$



Limits for the outer integral are the full range of y values anywhere in the region

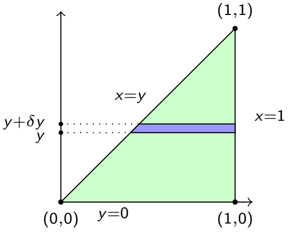
Triangular example — horizontal strips

$D =$ triangle with vertices $(0,0)$, $(1,0)$ and $(1,1)$.

$$\iint_D e^{2x-2y}\,dA = \int_{y=0}^{1}\left(\int_{x=y}^{1} e^{2x-2y}\,dx\right)dy$$

Limits in the inner integral are the range of x values for a particular y. In this integral, we treat y as a constant and x as a variable. This gives

$$\int_{x=y}^{1} e^{2x-2y}\,dx = \left[\tfrac{1}{2}e^{2x-2y}\right]_{x=y}^{1} = \tfrac{1}{2}(e^{2-2y}-1)$$



Limits for the outer integral are the full range of y values anywhere in the region, which means $0 \le y \le 1$ in this example.

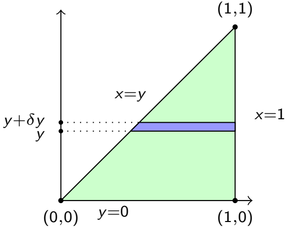
Triangular example — horizontal strips

$D =$ triangle with vertices $(0,0)$, $(1,0)$ and $(1,1)$.

$$\iint_D e^{2x-2y}\,dA = \int_{y=0}^{1}\left(\int_{x=y}^{1} e^{2x-2y}\,dx\right)dy$$

Limits in the inner integral are the range of x values for a particular y. In this integral, we treat y as a constant and x as a variable. This gives

$$\int_{x=y}^{1} e^{2x-2y}\,dx = \left[\tfrac{1}{2}e^{2x-2y}\right]_{x=y}^{1} = \tfrac{1}{2}(e^{2-2y}-1)$$



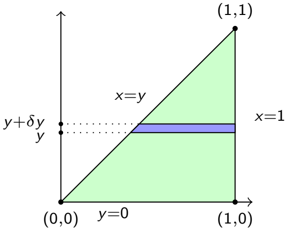
Limits for the outer integral are the full range of y values anywhere in the region, which means $0 \le y \le 1$ in this example.

$$\iint_D e^{2x-2y}\,dA = \int_{y=0}^{1} \tfrac{1}{2}(e^{2-2y}-1)\,dy$$

Triangular example — horizontal strips

$D =$ triangle with vertices $(0,0)$, $(1,0)$ and $(1,1)$.

$$\iint_D e^{2x-2y}\,dA = \int_{y=0}^{1}\left(\int_{x=y}^{1} e^{2x-2y}\,dx\right)dy$$



Limits in the inner integral are the range of x values for a particular y. In this integral, we treat y as a constant and x as a variable. This gives

$$\int_{x=y}^{1} e^{2x-2y}\,dx = \left[\tfrac{1}{2}e^{2x-2y}\right]_{x=y}^{1} = \tfrac{1}{2}(e^{2-2y}-1)$$

Limits for the outer integral are the full range of y values anywhere in the region, which means $0 \le y \le 1$ in this example.

$$\iint_D e^{2x-2y}\,dA = \int_{y=0}^{1} \tfrac{1}{2}(e^{2-2y}-1)\,dy = \left[\tfrac{1}{2}(-\tfrac{1}{2}e^{2-2y}-y)\right]_{y=0}^{1}$$

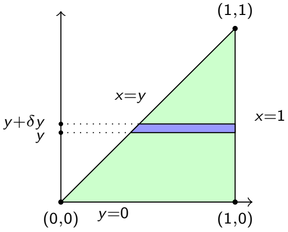
Triangular example — horizontal strips

$D =$ triangle with vertices $(0,0)$, $(1,0)$ and $(1,1)$.

$$\iint_D e^{2x-2y}\,dA = \int_{y=0}^{1}\left(\int_{x=y}^{1} e^{2x-2y}\,dx\right)dy$$

Limits in the inner integral are the range of x values for a particular y. In this integral, we treat y as a constant and x as a variable. This gives

$$\int_{x=y}^{1} e^{2x-2y}\,dx = \left[\tfrac{1}{2}e^{2x-2y}\right]_{x=y}^{1} = \tfrac{1}{2}(e^{2-2y}-1)$$



Limits for the outer integral are the full range of y values anywhere in the region, which means $0 \le y \le 1$ in this example.

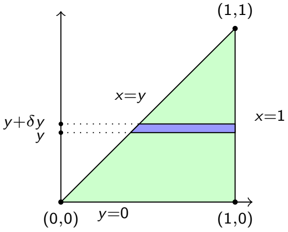
$$\iint_D e^{2x-2y}\,dA = \int_{y=0}^{1} \tfrac{1}{2}(e^{2-2y}-1)\,dy = \left[\tfrac{1}{2}(-\tfrac{1}{2}e^{2-2y}-y)\right]_{y=0}^{1}$$

$$= \tfrac{1}{2}(-\tfrac{1}{2}-1) - \tfrac{1}{2}(-\tfrac{1}{2}e^{2}-0)$$

Triangular example — horizontal strips

$D =$ triangle with vertices $(0,0)$, $(1,0)$ and $(1,1)$.

$$\iint_D e^{2x-2y}\,dA = \int_{y=0}^{1}\left(\int_{x=y}^{1} e^{2x-2y}\,dx\right)dy$$



Limits in the inner integral are the range of x values for a particular y. In this integral, we treat y as a constant and x as a variable. This gives

$$\int_{x=y}^{1} e^{2x-2y}\,dx = \left[\tfrac{1}{2}e^{2x-2y}\right]_{x=y}^{1} = \tfrac{1}{2}(e^{2-2y}-1)$$

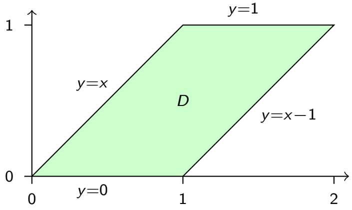
Limits for the outer integral are the full range of y values anywhere in the region, which means $0 \le y \le 1$ in this example.

$$\iint_D e^{2x-2y}\,dA = \int_{y=0}^{1} \tfrac{1}{2}(e^{2-2y}-1)\,dy = \left[\tfrac{1}{2}(-\tfrac{1}{2}e^{2-2y}-y)\right]_{y=0}^{1}$$

$$= \tfrac{1}{2}(-\tfrac{1}{2}-1) - \tfrac{1}{2}(-\tfrac{1}{2}e^2-0) = (e^2-3)/4.$$

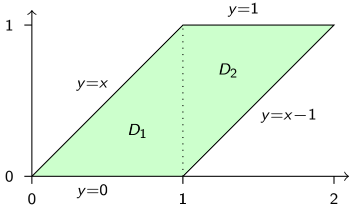
A split region

$$\iint_D xy \, dA$$



A split region

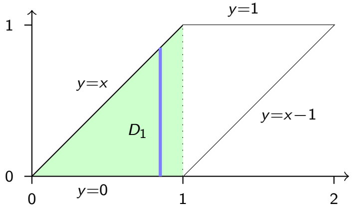
$$\iint_D xy\,dA = \iint_{D_1} xy\,dA + \iint_{D_2} xy\,dA$$



A split region

$$\iint_D xy\, dA = \iint_{D_1} xy\, dA + \iint_{D_2} xy\, dA$$

$$\iint_{D_1} xy\, dA = \int_{x=0}^{1} \int_{y=0}^{y=x} xy\, dy\, dx$$

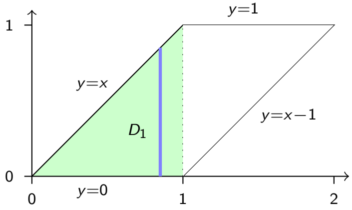


A split region

$$\iint_D xy\,dA = \iint_{D_1} xy\,dA + \iint_{D_2} xy\,dA$$

$$\iint_{D_1} xy\,dA = \int_{x=0}^{1}\int_{y=0}^{y=x} xy\,dy\,dx$$

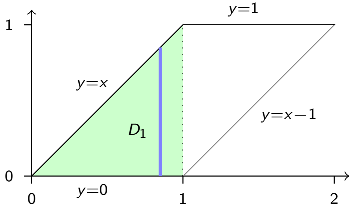
$$= \int_{x=0}^{1} \left[\tfrac{1}{2}xy^2\right]_{y=0}^{x}\,dx$$



A split region

$$\iint_D xy\,dA = \iint_{D_1} xy\,dA + \iint_{D_2} xy\,dA$$

$$\iint_{D_1} xy\,dA = \int_{x=0}^{1} \int_{y=0}^{y=x} xy\,dy\,dx$$

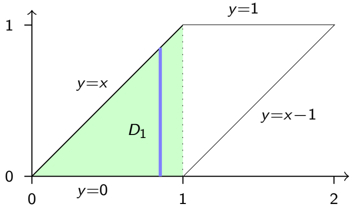


$$= \int_{x=0}^{1} \left[\tfrac{1}{2} xy^2 \right]_{y=0}^{x} dx = \int_{x=0}^{1} \tfrac{1}{2} x^3\,dx$$

A split region

$$\iint_D xy\,dA = \iint_{D_1} xy\,dA + \iint_{D_2} xy\,dA$$

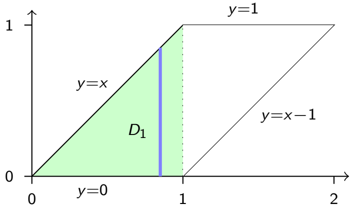
$$\iint_{D_1} xy\,dA = \int_{x=0}^{1} \int_{y=0}^{y=x} xy\,dy\,dx$$



$$= \int_{x=0}^{1} \left[\tfrac{1}{2}xy^2\right]_{y=0}^{x} dx = \int_{x=0}^{1} \tfrac{1}{2}x^3\,dx = \left[\tfrac{1}{8}x^4\right]_{x=0}^{1}$$

A split region

$$\iint_D xy\,dA = \iint_{D_1} xy\,dA + \iint_{D_2} xy\,dA$$



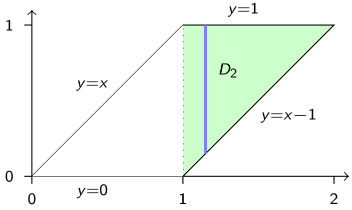
$$\iint_{D_1} xy\,dA = \int_{x=0}^{1} \int_{y=0}^{y=x} xy\,dy\,dx$$

$$= \int_{x=0}^{1} \left[\tfrac{1}{2}xy^2\right]_{y=0}^{x} dx = \int_{x=0}^{1} \tfrac{1}{2}x^3\,dx = \left[\tfrac{1}{8}x^4\right]_{x=0}^{1} = \frac{1}{8}.$$

A split region

$$\iint_D xy\,dA = \iint_{D_1} xy\,dA + \iint_{D_2} xy\,dA$$

$$\iint_{D_1} xy\,dA = \int_{x=0}^{1} \int_{y=0}^{y=x} xy\,dy\,dx$$

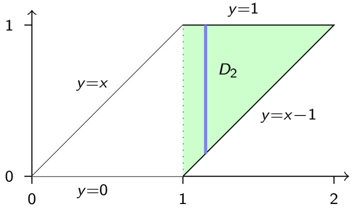


$$= \int_{x=0}^{1} \left[\tfrac{1}{2}xy^2\right]_{y=0}^{x} dx = \int_{x=0}^{1} \tfrac{1}{2}x^3\,dx = \left[\tfrac{1}{8}x^4\right]_{x=0}^{1} = \frac{1}{8}.$$

$$\iint_{D_2} xy\,dA = \int_{x=1}^{2} \int_{y=x-1}^{1} xy\,dy\,dx$$

A split region

$$\iint_D xy\, dA = \iint_{D_1} xy\, dA + \iint_{D_2} xy\, dA$$



$$\iint_{D_1} xy\, dA = \int_{x=0}^{1} \int_{y=0}^{y=x} xy\, dy\, dx$$

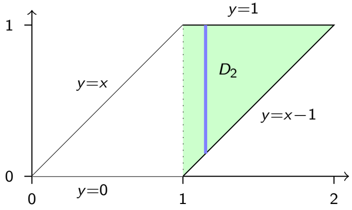
$$= \int_{x=0}^{1} \left[\tfrac{1}{2}xy^2\right]_{y=0}^{x} dx = \int_{x=0}^{1} \tfrac{1}{2}x^3\, dx = \left[\tfrac{1}{8}x^4\right]_{x=0}^{1} = \frac{1}{8}.$$

$$\iint_{D_2} xy\, dA = \int_{x=1}^{2} \int_{y=x-1}^{1} xy\, dy\, dx$$

$$\int_{y=x-1}^{1} xy\, dy$$

A split region

$$\iint_D xy\,dA = \iint_{D_1} xy\,dA + \iint_{D_2} xy\,dA$$



$$\iint_{D_1} xy\,dA = \int_{x=0}^{1} \int_{y=0}^{y=x} xy\,dy\,dx$$

$$= \int_{x=0}^{1} \left[\tfrac{1}{2}xy^2\right]_{y=0}^{x} dx = \int_{x=0}^{1} \tfrac{1}{2}x^3\,dx = \left[\tfrac{1}{8}x^4\right]_{x=0}^{1} = \frac{1}{8}.$$

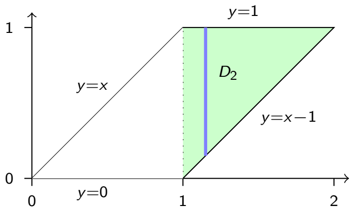
$$\iint_{D_2} xy\,dA = \int_{x=1}^{2} \int_{y=x-1}^{1} xy\,dy\,dx$$

$$\int_{y=x-1}^{1} xy\,dy = \left[\tfrac{1}{2}xy^2\right]_{y=x-1}^{1}$$

A split region

$$\iint_D xy\,dA = \iint_{D_1} xy\,dA + \iint_{D_2} xy\,dA$$

$$\iint_{D_1} xy\,dA = \int_{x=0}^{1} \int_{y=0}^{y=x} xy\,dy\,dx$$



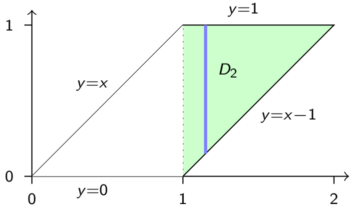
$$= \int_{x=0}^{1} \left[\tfrac{1}{2}xy^2\right]_{y=0}^{x} dx = \int_{x=0}^{1} \tfrac{1}{2}x^3\,dx = \left[\tfrac{1}{8}x^4\right]_{x=0}^{1} = \frac{1}{8}.$$

$$\iint_{D_2} xy\,dA = \int_{x=1}^{2} \int_{y=x-1}^{1} xy\,dy\,dx$$

$$\int_{y=x-1}^{1} xy\,dy = \left[\tfrac{1}{2}xy^2\right]_{y=x-1}^{1} = \frac{x}{2}\left(1^2 - (x-1)^2\right)$$

A split region

$$\iint_D xy\, dA = \iint_{D_1} xy\, dA + \iint_{D_2} xy\, dA$$



$$\iint_{D_1} xy\, dA = \int_{x=0}^{1} \int_{y=0}^{y=x} xy\, dy\, dx$$

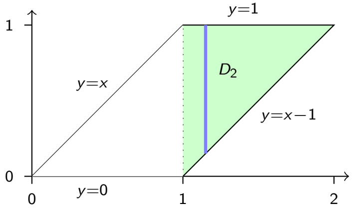
$$= \int_{x=0}^{1} \left[\tfrac{1}{2}xy^2\right]_{y=0}^{x} dx = \int_{x=0}^{1} \tfrac{1}{2}x^3\, dx = \left[\tfrac{1}{8}x^4\right]_{x=0}^{1} = \frac{1}{8}.$$

$$\iint_{D_2} xy\, dA = \int_{x=1}^{2} \int_{y=x-1}^{1} xy\, dy\, dx$$

$$\int_{y=x-1}^{1} xy\, dy = \left[\tfrac{1}{2}xy^2\right]_{y=x-1}^{1} = \frac{x}{2}\left(1^2 - (x-1)^2\right) = \frac{x}{2}\left(2x - x^2\right)$$

A split region

$$\iint_D xy\,dA = \iint_{D_1} xy\,dA + \iint_{D_2} xy\,dA$$



$$\iint_{D_1} xy\,dA = \int_{x=0}^{1}\int_{y=0}^{y=x} xy\,dy\,dx$$

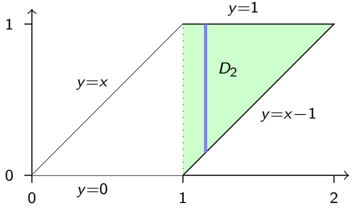
$$= \int_{x=0}^{1}\left[\tfrac{1}{2}xy^2\right]_{y=0}^{x} dx = \int_{x=0}^{1}\tfrac{1}{2}x^3\,dx = \left[\tfrac{1}{8}x^4\right]_{x=0}^{1} = \frac{1}{8}.$$

$$\iint_{D_2} xy\,dA = \int_{x=1}^{2}\int_{y=x-1}^{1} xy\,dy\,dx$$

$$\int_{y=x-1}^{1} xy\,dy = \left[\tfrac{1}{2}xy^2\right]_{y=x-1}^{1} = \frac{x}{2}\left(1^2-(x-1)^2\right) = \frac{x}{2}\left(2x-x^2\right) = x^2-\tfrac{1}{2}x^3.$$

A split region

$$\iint_D xy\,dA = \iint_{D_1} xy\,dA + \iint_{D_2} xy\,dA$$



$$\iint_{D_1} xy\,dA = \int_{x=0}^{1}\int_{y=0}^{y=x} xy\,dy\,dx$$

$$= \int_{x=0}^{1}\left[\tfrac{1}{2}xy^2\right]_{y=0}^{x} dx = \int_{x=0}^{1}\tfrac{1}{2}x^3\,dx = \left[\tfrac{1}{8}x^4\right]_{x=0}^{1} = \frac{1}{8}.$$

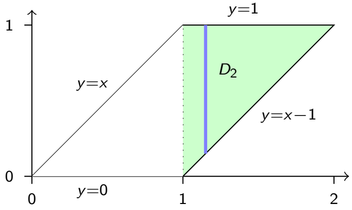
$$\iint_{D_2} xy\,dA = \int_{x=1}^{2}\int_{y=x-1}^{1} xy\,dy\,dx$$

$$\int_{y=x-1}^{1} xy\,dy = \left[\tfrac{1}{2}xy^2\right]_{y=x-1}^{1} = \frac{x}{2}\left(1^2-(x-1)^2\right) = \frac{x}{2}\left(2x-x^2\right) = x^2-\tfrac{1}{2}x^3.$$

$$\iint_{D_2} xy\,dA = \int_{x=1}^{2} x^2-\tfrac{1}{2}x^3\,dx$$

A split region

$$\iint_D xy\,dA = \iint_{D_1} xy\,dA + \iint_{D_2} xy\,dA$$



$$\iint_{D_1} xy\,dA = \int_{x=0}^{1} \int_{y=0}^{y=x} xy\,dy\,dx$$

$$= \int_{x=0}^{1} \left[\tfrac{1}{2}xy^2\right]_{y=0}^{x} dx = \int_{x=0}^{1} \tfrac{1}{2}x^3\,dx = \left[\tfrac{1}{8}x^4\right]_{x=0}^{1} = \frac{1}{8}.$$

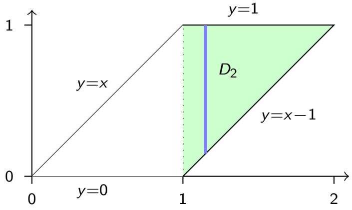
$$\iint_{D_2} xy\,dA = \int_{x=1}^{2} \int_{y=x-1}^{1} xy\,dy\,dx$$

$$\int_{y=x-1}^{1} xy\,dy = \left[\tfrac{1}{2}xy^2\right]_{y=x-1}^{1} = \frac{x}{2}\left(1^2 - (x-1)^2\right) = \frac{x}{2}\left(2x - x^2\right) = x^2 - \tfrac{1}{2}x^3.$$

$$\iint_{D_2} xy\,dA = \int_{x=1}^{2} x^2 - \tfrac{1}{2}x^3\,dx = \left[\frac{x^3}{3} - \frac{x^4}{8}\right]_{x=1}^{2}$$

A split region

$$\iint_D xy\,dA = \iint_{D_1} xy\,dA + \iint_{D_2} xy\,dA$$



$$\iint_{D_1} xy\,dA = \int_{x=0}^{1} \int_{y=0}^{y=x} xy\,dy\,dx$$

$$= \int_{x=0}^{1} \left[\tfrac{1}{2}xy^2\right]_{y=0}^{x} dx = \int_{x=0}^{1} \tfrac{1}{2}x^3\,dx = \left[\tfrac{1}{8}x^4\right]_{x=0}^{1} = \frac{1}{8}.$$

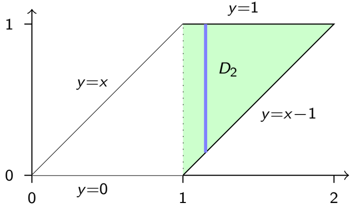
$$\iint_{D_2} xy\,dA = \int_{x=1}^{2} \int_{y=x-1}^{1} xy\,dy\,dx$$

$$\int_{y=x-1}^{1} xy\,dy = \left[\tfrac{1}{2}xy^2\right]_{y=x-1}^{1} = \frac{x}{2}\left(1^2 - (x-1)^2\right) = \frac{x}{2}\left(2x - x^2\right) = x^2 - \tfrac{1}{2}x^3.$$

$$\iint_{D_2} xy\,dA = \int_{x=1}^{2} x^2 - \tfrac{1}{2}x^3\,dx = \left[\frac{x^3}{3} - \frac{x^4}{8}\right]_{x=1}^{2} = \left(\tfrac{8}{3} - \tfrac{16}{8}\right) - \left(\tfrac{1}{3} - \tfrac{1}{8}\right)$$

A split region

$$\iint_D xy\,dA = \iint_{D_1} xy\,dA + \iint_{D_2} xy\,dA$$



$$\iint_{D_1} xy\,dA = \int_{x=0}^{1} \int_{y=0}^{y=x} xy\,dy\,dx$$

$$= \int_{x=0}^{1} \left[\tfrac{1}{2}xy^2\right]_{y=0}^{x} dx = \int_{x=0}^{1} \tfrac{1}{2}x^3\,dx = \left[\tfrac{1}{8}x^4\right]_{x=0}^{1} = \frac{1}{8}.$$

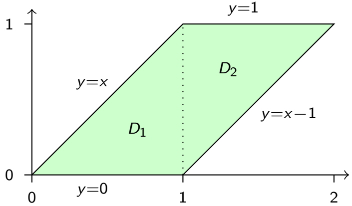
$$\iint_{D_2} xy\,dA = \int_{x=1}^{2} \int_{y=x-1}^{1} xy\,dy\,dx$$

$$\int_{y=x-1}^{1} xy\,dy = \left[\tfrac{1}{2}xy^2\right]_{y=x-1}^{1} = \frac{x}{2}\left(1^2 - (x-1)^2\right) = \frac{x}{2}\left(2x - x^2\right) = x^2 - \tfrac{1}{2}x^3.$$

$$\iint_{D_2} xy\,dA = \int_{x=1}^{2} x^2 - \tfrac{1}{2}x^3\,dx = \left[\frac{x^3}{3} - \frac{x^4}{8}\right]_{x=1}^{2} = \left(\tfrac{8}{3} - \tfrac{16}{8}\right) - \left(\tfrac{1}{3} - \tfrac{1}{8}\right) = 11/24.$$

A split region

$$\iint_D xy\,dA = \iint_{D_1} xy\,dA + \iint_{D_2} xy\,dA$$



$$\iint_{D_1} xy\,dA = \int_{x=0}^{1}\int_{y=0}^{y=x} xy\,dy\,dx$$

$$= \int_{x=0}^{1}\left[\tfrac{1}{2}xy^2\right]_{y=0}^{x} dx = \int_{x=0}^{1}\tfrac{1}{2}x^3\,dx = \left[\tfrac{1}{8}x^4\right]_{x=0}^{1} = \frac{1}{8}.$$

$$\iint_{D_2} xy\,dA = \int_{x=1}^{2}\int_{y=x-1}^{1} xy\,dy\,dx$$

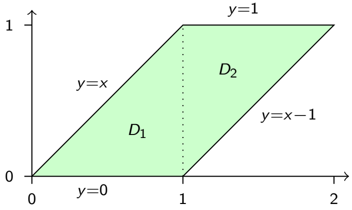
$$\int_{y=x-1}^{1} xy\,dy = \left[\tfrac{1}{2}xy^2\right]_{y=x-1}^{1} = \frac{x}{2}\left(1^2-(x-1)^2\right) = \frac{x}{2}\left(2x-x^2\right) = x^2-\tfrac{1}{2}x^3.$$

$$\iint_{D_2} xy\,dA = \int_{x=1}^{2} x^2-\tfrac{1}{2}x^3\,dx = \left[\frac{x^3}{3}-\frac{x^4}{8}\right]_{x=1}^{2} = \left(\tfrac{8}{3}-\tfrac{16}{8}\right)-\left(\tfrac{1}{3}-\tfrac{1}{8}\right) = 11/24.$$

$$\iint_D xy\,dA = \iint_{D_1} xy\,dA + \iint_{D_2} xy\,dA$$

A split region

$$\iint_D xy\,dA = \iint_{D_1} xy\,dA + \iint_{D_2} xy\,dA$$



$$\iint_{D_1} xy\,dA = \int_{x=0}^{1} \int_{y=0}^{y=x} xy\,dy\,dx$$

$$= \int_{x=0}^{1} \left[\tfrac{1}{2}xy^2\right]_{y=0}^{x} dx = \int_{x=0}^{1} \tfrac{1}{2}x^3\,dx = \left[\tfrac{1}{8}x^4\right]_{x=0}^{1} = \frac{1}{8}.$$

$$\iint_{D_2} xy\,dA = \int_{x=1}^{2} \int_{y=x-1}^{1} xy\,dy\,dx$$

$$\int_{y=x-1}^{1} xy\,dy = \left[\tfrac{1}{2}xy^2\right]_{y=x-1}^{1} = \frac{x}{2}\left(1^2 - (x-1)^2\right) = \frac{x}{2}\left(2x - x^2\right) = x^2 - \tfrac{1}{2}x^3.$$

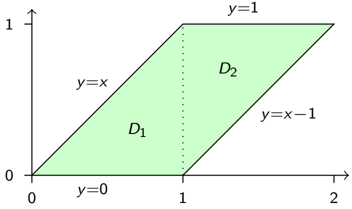
$$\iint_{D_2} xy\,dA = \int_{x=1}^{2} x^2 - \tfrac{1}{2}x^3\,dx = \left[\frac{x^3}{3} - \frac{x^4}{8}\right]_{x=1}^{2} = (\tfrac{8}{3} - \tfrac{16}{8}) - (\tfrac{1}{3} - \tfrac{1}{8}) = 11/24.$$

$$\iint_D xy\,dA = \iint_{D_1} xy\,dA + \iint_{D_2} xy\,dA = \frac{1}{8} + \frac{11}{24}$$

A split region

$$\iint_D xy\,dA = \iint_{D_1} xy\,dA + \iint_{D_2} xy\,dA$$

$$\iint_{D_1} xy\,dA = \int_{x=0}^{1}\int_{y=0}^{y=x} xy\,dy\,dx$$



$$= \int_{x=0}^{1}\left[\tfrac{1}{2}xy^2\right]_{y=0}^{x} dx = \int_{x=0}^{1}\tfrac{1}{2}x^3\,dx = \left[\tfrac{1}{8}x^4\right]_{x=0}^{1} = \frac{1}{8}.$$

$$\iint_{D_2} xy\,dA = \int_{x=1}^{2}\int_{y=x-1}^{1} xy\,dy\,dx$$

$$\int_{y=x-1}^{1} xy\,dy = \left[\tfrac{1}{2}xy^2\right]_{y=x-1}^{1} = \frac{x}{2}\left(1^2-(x-1)^2\right) = \frac{x}{2}\left(2x-x^2\right) = x^2-\tfrac{1}{2}x^3.$$

$$\iint_{D_2} xy\,dA = \int_{x=1}^{2} x^2-\tfrac{1}{2}x^3\,dx = \left[\frac{x^3}{3}-\frac{x^4}{8}\right]_{x=1}^{2} = \left(\tfrac{8}{3}-\tfrac{16}{8}\right)-\left(\tfrac{1}{3}-\tfrac{1}{8}\right) = 11/24.$$

$$\iint_D xy\,dA = \iint_{D_1} xy\,dA + \iint_{D_2} xy\,dA = \frac{1}{8}+\frac{11}{24} = \frac{7}{12}.$$