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(c) Total mass of a star: integrate the density.

(d) Centre of mass $(\overline{x}, \overline{y}, \overline{z})$ of an object: $\overline{x} = \left(\iiint_E x\, dV\right) / \left(\iiint_E 1\, dV\right)$ and similarly for $\overline{y}$ and $\overline{z}$.

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Microwave oven

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We can integrate over y and then over x in the same way, giving

$$I = \int_{x=0}^{a} \sin^2(k\pi x)\frac{b}{2}\frac{c}{2}\, dx. = \frac{a}{2}.\frac{b}{2}.\frac{c}{2} = \frac{abc}{8}.$$

Moment of inertia of a cube

Let E be the cube given by $-1 \leq x, y, z \leq 1$.

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\iiint_E (x^2 + y^2)dV &= \int_{x=-1}^{1} \int_{y=-1}^{1} \int_{z=-1}^{1} (x^2 + y^2)dz\,dy\,dx \\
&= \int_{x=-1}^{1} \int_{y=-1}^{1} \left[x^2 z + y^2 z\right]_{z=-1}^{1} dy\,dx \\
&= \int_{x=-1}^{1} \int_{y=-1}^{1} 2(x^2 + y^2)dy\,dx
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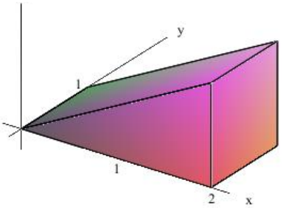
&= \left[4x^3/3 + 4x/3 \right]_{x=-1}^{1} = 8/3 + 8/3 = 16/3.
\end{aligned}$$

Mass centre of a wedge (constant density)



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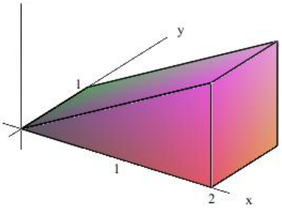

Limits are $0 \leq x \leq 2$

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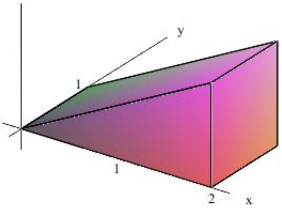

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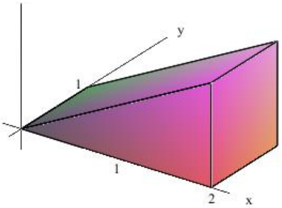

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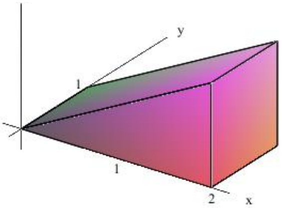

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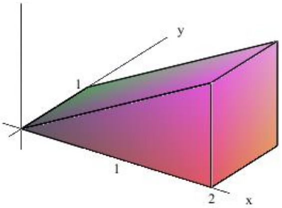

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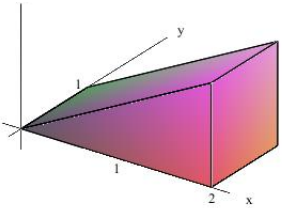

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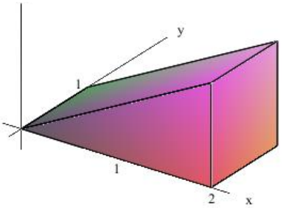

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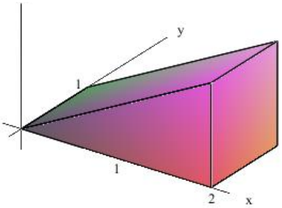

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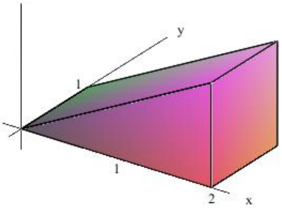

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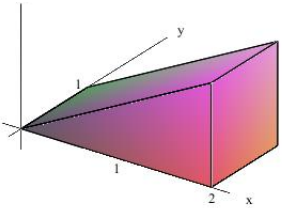

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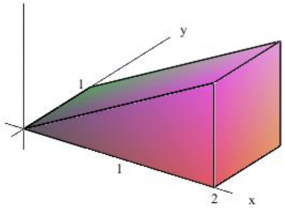

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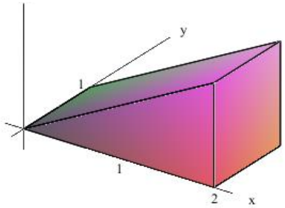

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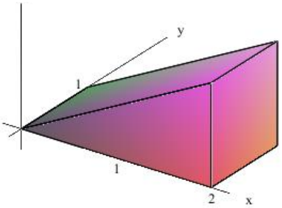

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$$= \left[\frac{x^3}{6}\right]_{x=0}^{2} = \frac{4}{3}$$

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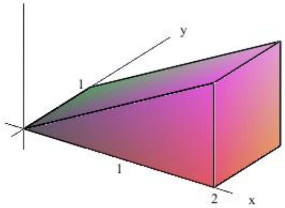
Mass centre of a wedge (constant density)



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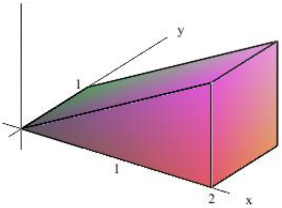

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$$Z = \iiint_E z \, dV$$

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$$\begin{aligned}
Z &= \iiint_E z\,dV = \int_{x=0}^{2}\int_{y=0}^{1}\int_{z=0}^{x/2} z\,dz\,dy\,dx = \int_{x=0}^{2}\int_{y=0}^{1}\left[\frac{z^2}{2}\right]_{z=0}^{x/2} dy\,dx \\
&= \int_{x=0}^{2}\int_{y=0}^{1}\frac{x^2}{8}\,dy\,dx = \int_{x=0}^{2}\frac{x^2}{8}\,dx = \left[\frac{x^3}{24}\right]_{x=0}^{2} = \frac{1}{3} \\
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&= \int_{x=0}^{2}\frac{x}{2}\,dx = \left[\frac{x^2}{4}\right]_{x=0}^{2} = 1
\end{aligned}$$

The centre of mass is

$$(\overline{x}, \overline{y}, \overline{z}) = (X/M, Y/M, Z/M)$$

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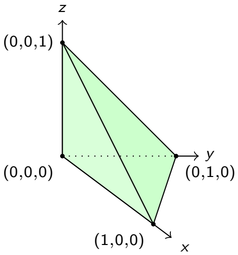
Integral over a tetrahedron

Let E be the tetrahedron with vertices $(0, 0, 0)$, $(1, 0, 0)$, $(0, 1, 0)$ and $(0, 0, 1)$.



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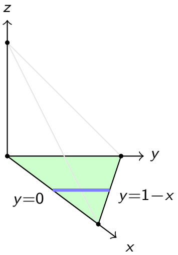

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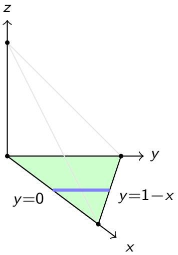

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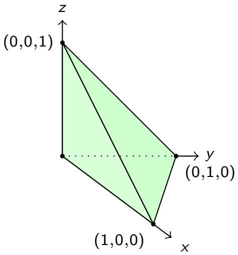

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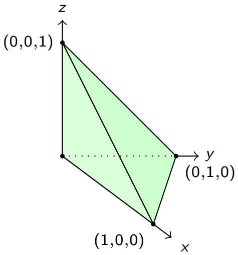

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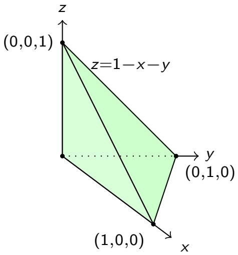

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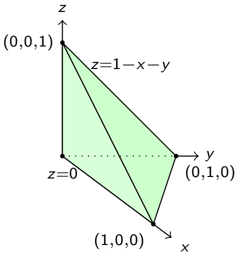

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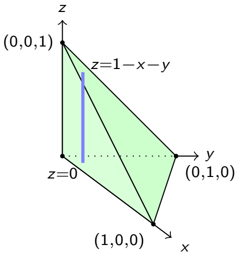

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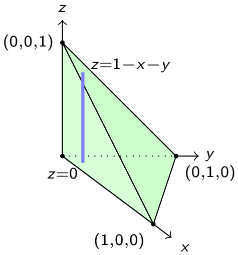

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$$\iiint_E f(x,y,z)\,dV = \int_{x=0}^{1}\int_{y=0}^{1-x}\int_{z=0}^{1-x-y} f(x,y,z)\,dz\,dy\,dx.$$

Volume of a tetrahedron

For a tetrahedron E with vertices $(0,0,0)$, $(1,0,0)$, $(0,1,0)$ and $(0,0,1)$:

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For the volume of E: take $f(x,y,z) = 1$.

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$$\int_{z=0}^{1-x-y} 1\,dz = 1 - x - y.$$

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Thus, the integral with respect to y is

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$$\int_{y=0}^{1-x}(1-x-y)\,dy = \Big[(1-x)y - y^2/2\Big]_{y=0}^{1-x} = ((1-x)(1-x)-(1-x)^2/2)-0$$
$$= 1/2 - x + x^2/2.$$

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For the volume of E: take $f(x,y,z) = 1$. The innermost integral is then

$$\int_{z=0}^{1-x-y} 1\,dz = 1 - x - y.$$

Thus, the integral with respect to y is

$$\int_{y=0}^{1-x} (1-x-y)\,dy = \left[(1-x)y - y^2/2\right]_{y=0}^{1-x} = ((1-x)(1-x) - (1-x)^2/2) - 0$$

$$= 1/2 - x + x^2/2.$$

Finally, the outermost integral (with respect to x) is

$$\int_{x=0}^{1} (1/2 - x + x^2/2)\,dx$$

Volume of a tetrahedron

For a tetrahedron E with vertices $(0,0,0)$, $(1,0,0)$, $(0,1,0)$ and $(0,0,1)$:

$$\iiint_E f(x,y,z)\,dV = \int_{x=0}^{1}\int_{y=0}^{1-x}\int_{z=0}^{1-x-y} f(x,y,z)\,dz\,dy\,dx.$$

For the volume of E: take $f(x,y,z)=1$. The innermost integral is then

$$\int_{z=0}^{1-x-y} 1\,dz = 1-x-y.$$

Thus, the integral with respect to y is

$$\int_{y=0}^{1-x} (1-x-y)\,dy = \Big[(1-x)y - y^2/2\Big]_{y=0}^{1-x} = ((1-x)(1-x)-(1-x)^2/2)-0$$
$$= 1/2 - x + x^2/2.$$

Finally, the outermost integral (with respect to x) is

$$\int_{x=0}^{1} (1/2 - x + x^2/2)\,dx = \Big[x/2 - x^2/2 + x^3/6\Big]_{x=0}^{1}$$

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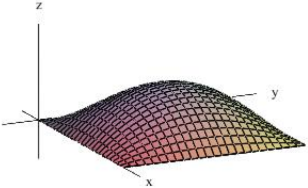
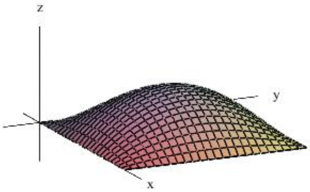
We conclude that the volume of the tetrahedron is $1/6$.

A curved region

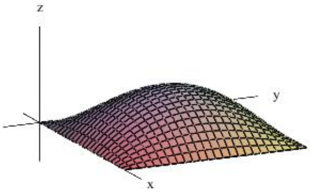


A curved region



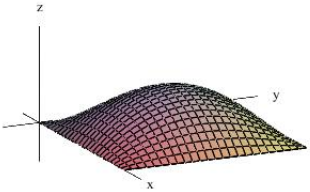
The base is the square where $0 \leq x, y \leq 1$ and $z = 0$.

A curved region



The base is the square where $0 \leq x, y \leq 1$ and $z = 0$.
The top surface is given by $z = 4(x - x^2)(y - y^2)$.

A curved region



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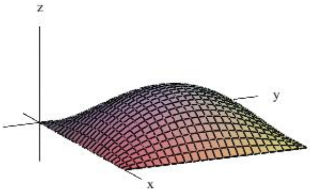
The volume is

$$V = \iiint_E 1 \, dV$$

A curved region



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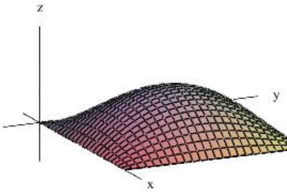
$$V = \iiint_E 1\, dV = \int_{x=0}^{1} \int_{y=0}^{1} \int_{z=0}^{4(x-x^2)(y-y^2)} 1\, dz\, dx\, dy$$

A curved region



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The volume is

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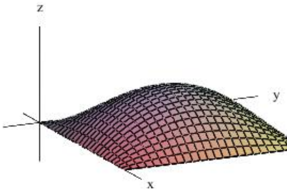
$$= \int_{x=0}^{1} \int_{y=0}^{1} 4(x - x^2)(y - y^2)\, dy\, dx$$

A curved region



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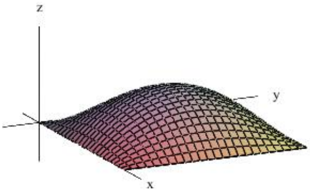
$$= \int_{x=0}^{1} \int_{y=0}^{1} 4(x - x^2)(y - y^2)\, dy\, dx = 4 \left(\int_{x=0}^{1} x - x^2\, dx \right) \left(\int_{y=0}^{1} y - y^2\, dy \right)$$

A curved region



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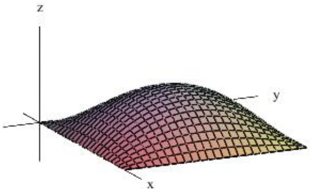
$$= \int_{x=0}^{1} \int_{y=0}^{1} 4(x - x^2)(y - y^2)\, dy\, dx = 4 \left(\int_{x=0}^{1} x - x^2\, dx \right) \left(\int_{y=0}^{1} y - y^2\, dy \right)$$

$$= 4 \left[\frac{x^2}{2} - \frac{x^3}{3} \right]_{x=0}^{1} \left[\frac{y^2}{2} - \frac{y^3}{3} \right]_{y=0}^{1}$$

A curved region



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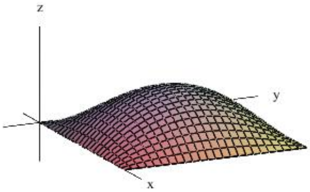
A curved region



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The volume is

$$V = \iiint_E 1\, dV = \int_{x=0}^{1} \int_{y=0}^{1} \int_{z=0}^{4(x-x^2)(y-y^2)} 1\, dz\, dx\, dy$$

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$$= 4 \left[\frac{x^2}{2} - \frac{x^3}{3} \right]_{x=0}^{1} \left[\frac{y^2}{2} - \frac{y^3}{3} \right]_{y=0}^{1} = 4 \times \frac{1}{6} \times \frac{1}{6} = \frac{1}{9}.$$