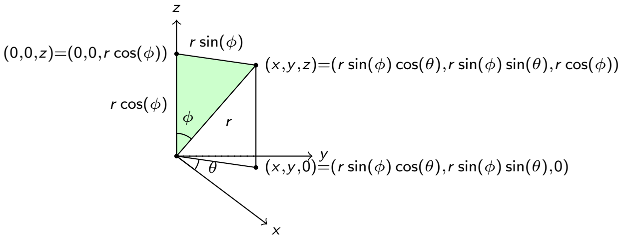
Spherical polar coordinates

To be read together with
http://shef.ac.uk/nps/courses/MAS243/pics/sph_polar.html

Spherical polar coordinates

In spherical polar coordinates we describe a point (x, y, z) by giving the distance r from the origin, the angle θ anticlockwise from the xz plane, and the angle ϕ from the z-axis.



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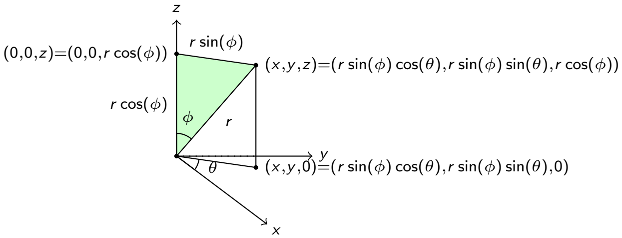

The variables r, θ and ϕ are related to x and y by the equations

$$x = r\sin(\phi)\cos(\theta) \qquad y = r\sin(\phi)\sin(\theta) \qquad z = r\cos(\phi)$$

$$r = \sqrt{x^2 + y^2 + z^2} \qquad \theta = \arctan(y/x) \qquad \phi = \arctan(\sqrt{x^2 + y^2}/z).$$

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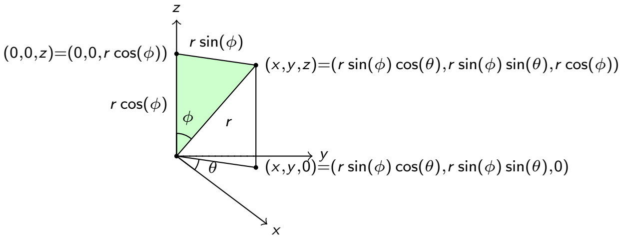

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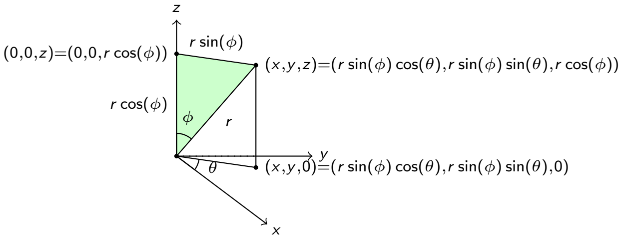

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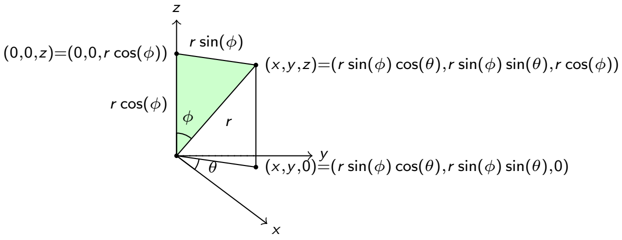

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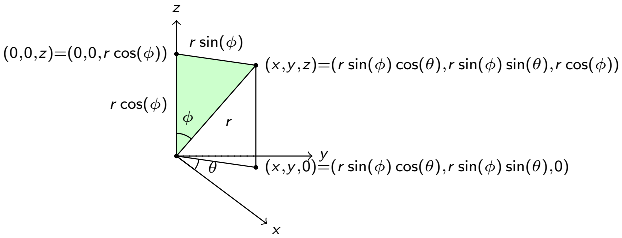

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Note that ϕ ranges from 0 (on the positive z-axis) to π (on the negative z-axis), whereas θ ranges from 0 to 2π (or equivalently, from $-\pi$ to π). It is also useful to observe that $\sqrt{x^2+y^2} = r\sin(\phi)$.

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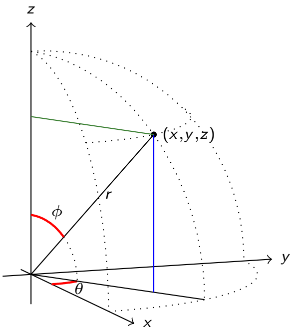

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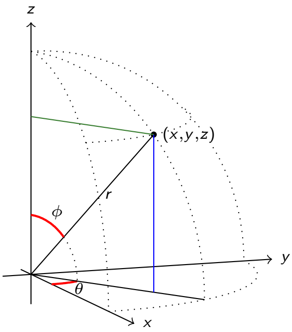

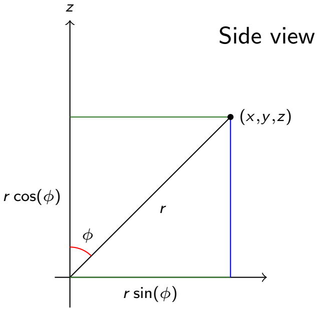

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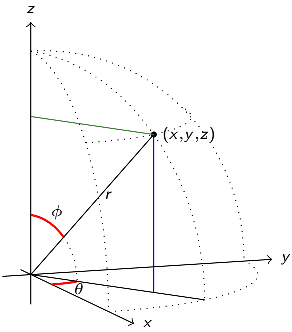
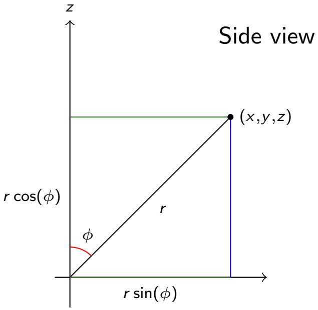
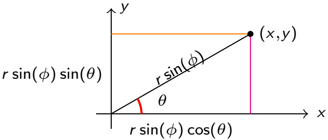
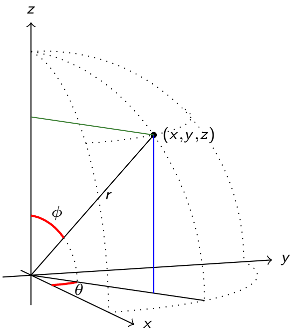


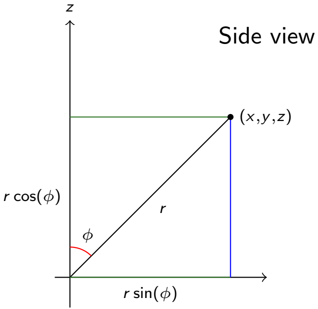


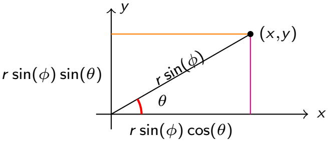


Top view

Spherical polar coordinates







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$x = r \sin(\phi) \cos(\theta)$

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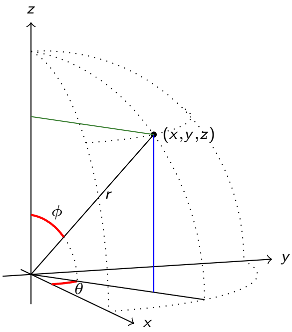

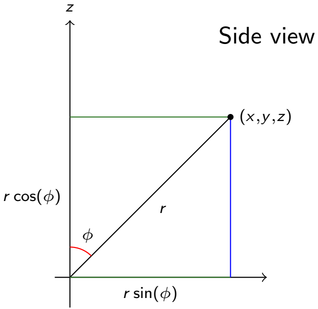

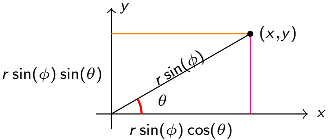

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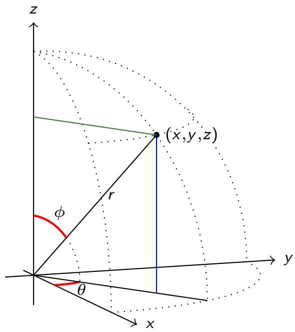

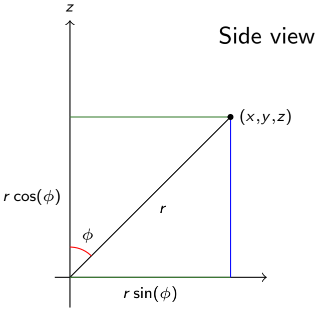

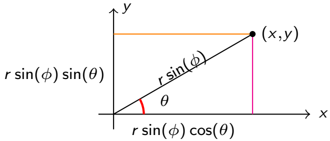

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Spherical polar volume element

For these coordinates it is easiest to find the area element using the Jacobian. We have $x = r\sin(\phi)\cos(\theta)$, $y = r\sin(\phi)\sin(\theta)$, $z = r\cos(\phi)$ so

$$J = \frac{\partial(x,y,z)}{\partial(r,\theta,\phi)} = \begin{bmatrix} x_r & x_\theta & x_\phi \\ y_r & y_\theta & y_\phi \\ z_r & z_\theta & z_\phi \end{bmatrix} = \begin{bmatrix} \sin(\phi)\cos(\theta) & -r\sin(\phi)\sin(\theta) & r\cos(\phi)\cos(\theta) \\ \sin(\phi)\sin(\theta) & r\sin(\phi)\cos(\theta) & r\cos(\phi)\sin(\theta) \\ \cos(\phi) & 0 & -r\sin(\phi) \end{bmatrix}$$

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As $0 \le \phi \le \pi$ we have $\sin(\phi) \ge 0$ so $|-r^2\sin(\phi)| = r^2\sin(\phi)$.

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As $0 \leq \phi \leq \pi$ we have $\sin(\phi) \geq 0$ so $|-r^2\sin(\phi)| = r^2\sin(\phi)$. We conclude that

$$dV = |\det(J)|\,dr\,d\theta\,d\phi = r^2\sin(\phi)\,dr\,d\theta\,d\phi.$$

Spherical polar volume element

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This means that for a function f on a 3-dimensional region E, we have

$$\iiint_E f(x,y,z)\,dV =$$

$$\int_{\phi=\cdots}^{\cdots}\int_{\theta=\cdots}^{\cdots}\int_{r=\cdots}^{\cdots} f(r\cos(\theta)\sin(\phi), r\sin(\theta)\sin(\phi), r\cos(\phi))\,r^2\sin(\phi)\,dr\,d\theta\,d\phi,$$

where the limits must be determined using the geometry of the region.

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The volume of a sphere E of radius a is

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Here the three different variables do not interact in any interesting way so we can rewrite the integral as

$$I = \left(\int_{\phi=0}^{\pi} \sin(\phi)^3\, d\phi \right) \left(\int_{\theta=0}^{2\pi} 1\, d\theta \right) \left(\int_{r=0}^{a} r^4\, dr \right) \rho.$$

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Two of these integrals are easy: we have $\int_{\theta=0}^{2\pi} 1 \, d\theta = 2\pi$ and $\int_{r=0}^{a} r^4 \, dr = a^5/5$.

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$$\sin^3(\phi) = \frac{1}{8j^3}(e^{3j\phi} - 3e^{2j\phi}e^{-j\phi} + 3e^{j\phi}e^{-2j\phi} - e^{-3j\phi})$$

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Volume and moment of a sphere

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\end{aligned}$$

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Combining this with the r and θ integrals gives $I = \frac{4}{3}.2\pi.\frac{a^5}{5}.\rho$

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Two of these integrals are easy: we have $\int_{\theta=0}^{2\pi} 1 \, d\theta = 2\pi$ and $\int_{r=0}^{a} r^4 \, dr = a^5/5$. For the integral with respect to ϕ, we recall that $\sin(\phi) = (e^{j\phi} - e^{-j\phi})/(2j)$. We can cube this to get

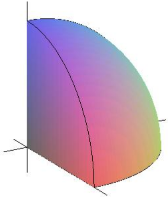
$$\begin{aligned}
\sin^3(\phi) &= \frac{1}{8j^3}(e^{3j\phi} - 3e^{2j\phi}e^{-j\phi} + 3e^{j\phi}e^{-2j\phi} - e^{-3j\phi}) \\
&= \frac{-1}{8j}(e^{3j\phi} - 3e^{j\phi} + 3e^{-j\phi} - e^{-3j\phi}) \\
&= \frac{3}{4}\left(\frac{e^{j\phi} - e^{-j\phi}}{2j}\right) - \frac{1}{4}\left(\frac{e^{3j\phi} - e^{-3j\phi}}{2j}\right) = \frac{3}{4}\sin(\phi) - \frac{1}{4}\sin(3\phi).
\end{aligned}$$

$$\begin{aligned}
\int_{\phi=0}^{\pi} \sin^3(\phi) \, d\phi &= \left[-\frac{3}{4}\cos(\phi) + \frac{1}{12}\cos(3\phi)\right]_{\phi=0}^{\pi} \\
&= (3/4 - 1/12) - (-3/4 + 1/12) = 4/3.
\end{aligned}$$

Combining this with the r and θ integrals gives $I = \frac{4}{3}.2\pi.\frac{a^5}{5}.\rho = \frac{8\pi a^5 \rho}{15}$.

Mass centre of an octant

Let E be the part of a sphere of radius 1 where $x \geq 0$, $y \geq 0$ and $z \geq 0$.



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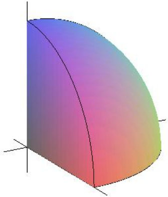

The centre of mass of E (assuming constant density) is $(\overline{x}, \overline{y}, \overline{z})$, where $\overline{x} = (\int\int\int_E x \, dV)/(\int\int\int_E 1 \, dV)$ and so on.

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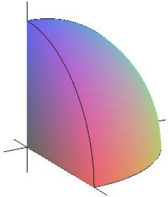

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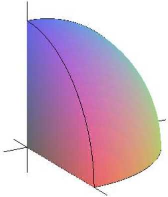

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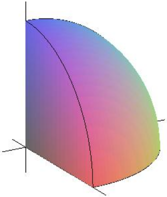

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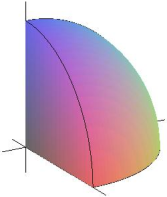

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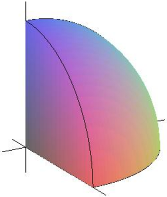

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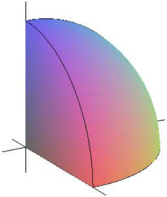

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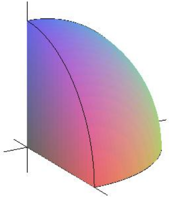

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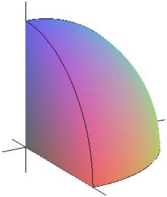

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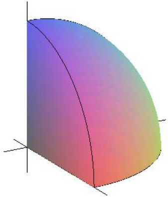

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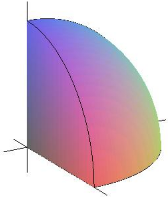

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This gives

$$Z = \int_{\phi=0}^{\frac{\pi}{2}} \int_{\theta=0}^{\frac{\pi}{2}} \int_{r=0}^{1} \tfrac{1}{2} r^3 \sin(2\phi)\, dr\, d\theta\, d\phi$$

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&= \frac{1}{2} \left(\int_{\phi=0}^{\frac{\pi}{2}} \sin(2\phi)\, d\phi \right) \left(\int_{\theta=0}^{\frac{\pi}{2}} 1\, d\theta \right) \left(\int_{r=0}^{1} r^3\, dr \right)
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so

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\bar{z} = Z/V = \frac{\pi}{16} \Big/ \frac{\pi}{6}
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$$

We conclude that the centre of mass is $(\frac{3}{8}, \frac{3}{8}, \frac{3}{8})$.