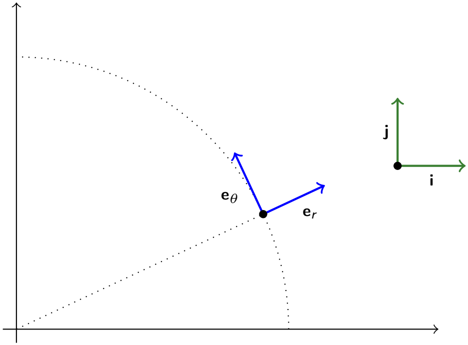
Vector fields in polar coordinates

Two dimensions

At any point in the plane, we can define vectors $\mathbf{r}_r$ and $\mathbf{e}_\theta$ as shown:



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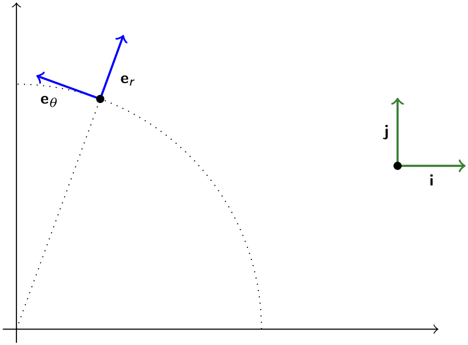

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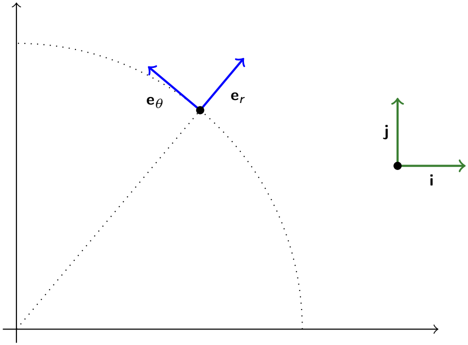

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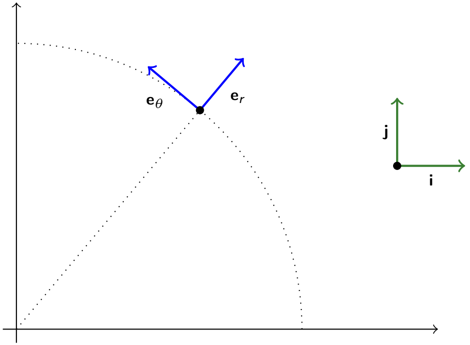

In situations with circular symmetry, it is often more natural to describe vector fields in terms of $\mathbf{e}_r$ and $\mathbf{e}_\theta$ rather than $\mathbf{i}$ and $\mathbf{j}$.

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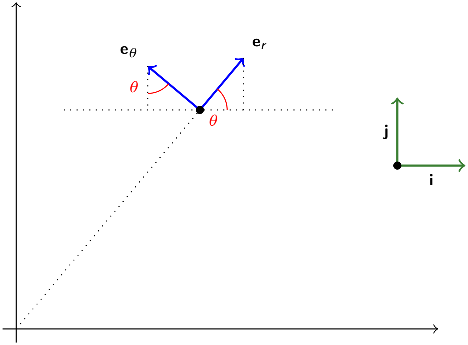

In situations with circular symmetry, it is often more natural to describe vector fields in terms of $\mathbf{e}_r$ and $\mathbf{e}_\theta$ rather than $\mathbf{i}$ and $\mathbf{j}$. One can translate between the two descriptions as follows:

$$\mathbf{e}_r = \cos(\theta)\mathbf{i} + \sin(\theta)\mathbf{j} \qquad \mathbf{e}_\theta = -\sin(\theta)\mathbf{i} + \cos(\theta)\mathbf{j}$$

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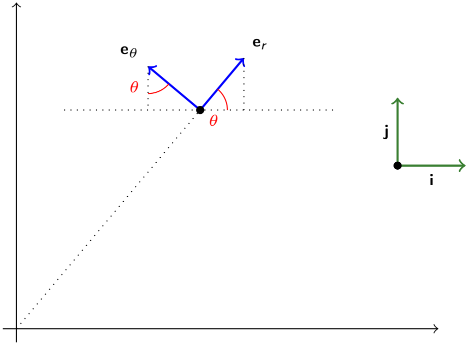
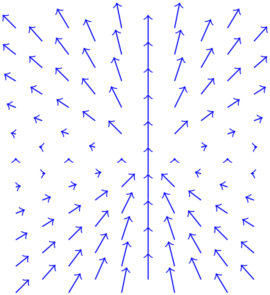


In situations with circular symmetry, it is often more natural to describe vector fields in terms of $\mathbf{e}_r$ and $\mathbf{e}_\theta$ rather than $\mathbf{i}$ and $\mathbf{j}$. One can translate between the two descriptions as follows:

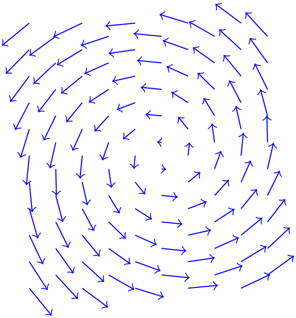
$$\mathbf{e}_r = \cos(\theta)\mathbf{i} + \sin(\theta)\mathbf{j} \qquad \mathbf{e}_\theta = -\sin(\theta)\mathbf{i} + \cos(\theta)\mathbf{j}$$

$$\mathbf{i} = \cos(\theta)\mathbf{e}_r - \sin(\theta)\mathbf{e}_\theta \qquad \mathbf{j} = \sin(\theta)\mathbf{e}_r + \cos(\theta)\mathbf{e}_\theta.$$

Here are two examples of vector fields described in terms of $\mathbf{e}_r$ and $\mathbf{e}_\theta$:



$\mathbf{u} = \sin(\theta)\mathbf{e}_r$



$\mathbf{u} = \sqrt{r}(\mathbf{e}_\theta + \mathbf{e}_r/10)$

Div, grad and curl in polar coordinates

We will need to express the operators grad, div and curl in terms of polar coordinates.

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Note: in the exam, if you need these formulae, they will be provided.

Grad in polar coordinates

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Examples of polar div, grad and curl

Example: Consider $f = r^n$. Clearly $f_r = nr^{n-1}$ and $f_\theta = 0$, so

$$\text{grad}(f) = f_r\,\mathbf{e}_r + r^{-1} f_\theta\,\mathbf{e}_\theta = nr^{n-1}\mathbf{e}_r.$$

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Example: Consider $f = \theta$. Clearly $f_r = 0$ and $f_\theta = 1$, so

$$\text{grad}(f) = f_r\,\mathbf{e}_r + r^{-1} f_\theta\,\mathbf{e}_\theta = r^{-1}\mathbf{e}_\theta = r^{-2}(-r\sin(\theta), r\cos(\theta)) = \left(\frac{-y}{x^2+y^2}, \frac{x}{x^2+y^2}\right).$$

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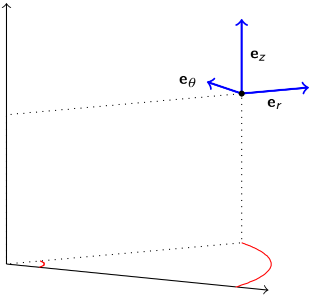
Cylindrical polar coordinates

In cylindrical polar coordinates we use unit vectors $\mathbf{e}_r$, $\mathbf{e}_\theta$ and $\mathbf{e}_z$ as shown below:



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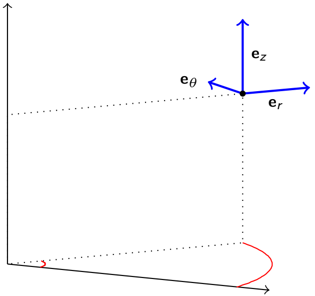

Thus, $\mathbf{e}_r$ and $\mathbf{e}_\theta$ are the same as for two-dimensional polar coordinates, and $\mathbf{e}_z$ is just the vertical unit vector $\mathbf{k}$.

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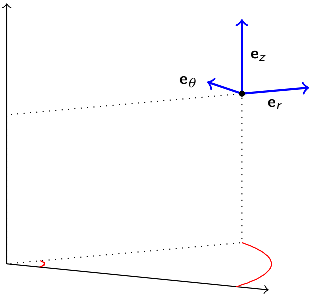

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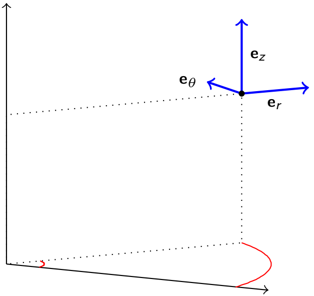

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$$\mathbf{i} = \cos(\theta)\mathbf{e}_r - \sin(\theta)\mathbf{e}_\theta \qquad \mathbf{j} = \sin(\theta)\mathbf{e}_r + \cos(\theta)\mathbf{e}_\theta \qquad \mathbf{k} = \mathbf{e}_z.$$

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Div, grad and curl in cylindrical polar coordinates

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(a) For any three-dimensional scalar field f (expressed as a function of r, θ and z) we have

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Consider the vector field $\mathbf{u}$ given in cylindrical polar coordinates by $\mathbf{u} = r(\mathbf{e}_\theta + \mathbf{e}_z)$.

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$$\text{curl}(\mathbf{u})$$

$$= \frac{1}{r} \det \begin{bmatrix} \mathbf{e}_r & r\mathbf{e}_\theta & \mathbf{e}_z \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial \theta} & \frac{\partial}{\partial z} \\ 0 & r^2 & r \end{bmatrix}$$

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$$\begin{aligned}
&\operatorname{curl}(\mathbf{u}) \\
&= \frac{1}{r}\det\begin{bmatrix} \mathbf{e}_r & r\mathbf{e}_\theta & \mathbf{e}_z \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial \theta} & \frac{\partial}{\partial z} \\ 0 & r^2 & r \end{bmatrix} \\
&= \frac{1}{r}\left(\left(\frac{\partial}{\partial \theta}(r) - \frac{\partial}{\partial z}(r^2)\right)\mathbf{e}_r - \left(\frac{\partial}{\partial r}(r) - \frac{\partial}{\partial z}(0)\right) r\mathbf{e}_\theta + \left(\frac{\partial}{\partial r}(r^2) - \frac{\partial}{\partial \theta}(0)\right)\mathbf{e}_z\right)
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$$
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&= \frac{1}{r}\left(\left(\frac{\partial}{\partial \theta}(r) - \frac{\partial}{\partial z}(r^2) \right) \mathbf{e}_r - \left(\frac{\partial}{\partial r}(r) - \frac{\partial}{\partial z}(0) \right) r\mathbf{e}_\theta + \left(\frac{\partial}{\partial r}(r^2) - \frac{\partial}{\partial \theta}(0) \right) \mathbf{e}_z \right) \\
&= \frac{1}{r}\left(-r\mathbf{e}_\theta + 2r\mathbf{e}_z\right)
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Consider the vector field $\mathbf{u}$ given in cylindrical polar coordinates by $\mathbf{u} = r(\mathbf{e}_\theta + \mathbf{e}_z)$. This is $\mathbf{u} = m\mathbf{e}_r + p\mathbf{e}_\theta + q\mathbf{e}_z$, where $m = 0$ and $p = q = r$, so

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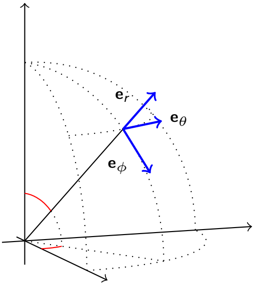
&= \frac{1}{r} \left(-r\mathbf{e}_\theta + 2r\mathbf{e}_z \right) = 2\mathbf{e}_z - \mathbf{e}_\theta .
\end{aligned}
$$

Spherical polar coordinates

In spherical polar coordinates we use unit vectors $\mathbf{e}_r$, $\mathbf{e}_\theta$ and $\mathbf{e}_\phi$ as on the right:



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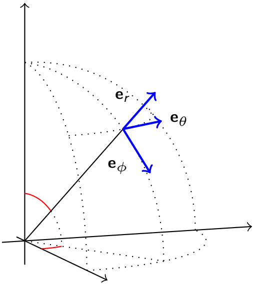

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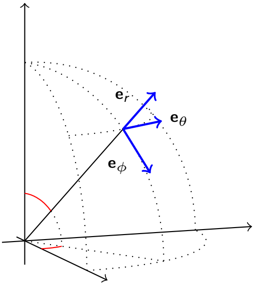
Note that $\mathbf{e}_\theta$ has the same meaning as it did in the cylindrical case, but $\mathbf{e}_r$ has changed. It used to be the unit vector pointing horizontally away from the z-axis, but now it points directly away from the origin.



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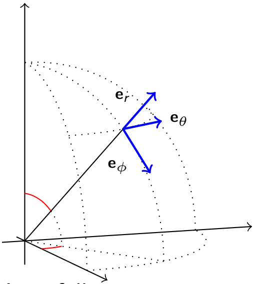

The vectors $\mathbf{e}_r$, $\mathbf{e}_\phi$ and $\mathbf{e}_\theta$ are related to $\mathbf{i}$, $\mathbf{j}$ and $\mathbf{k}$ as follows.

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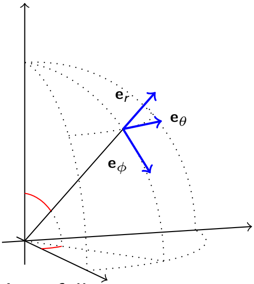

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$$\begin{aligned}
\mathbf{e}_r &= \sin(\phi)\cos(\theta)\mathbf{i} + \sin(\phi)\sin(\theta)\mathbf{j} + \cos(\phi)\mathbf{k} \\
\mathbf{e}_\phi &= \cos(\phi)\cos(\theta)\mathbf{i} + \cos(\phi)\sin(\theta)\mathbf{j} - \sin(\phi)\mathbf{k} \\
\mathbf{e}_\theta &= -\sin(\theta)\mathbf{i} + \cos(\theta)\mathbf{j}
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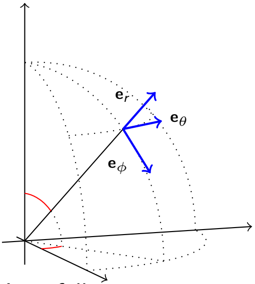

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\begin{aligned}
\mathbf{e}_r &= \sin(\phi)\cos(\theta)\mathbf{i} + \sin(\phi)\sin(\theta)\mathbf{j} + \cos(\phi)\mathbf{k} \\
\mathbf{e}_\phi &= \cos(\phi)\cos(\theta)\mathbf{i} + \cos(\phi)\sin(\theta)\mathbf{j} - \sin(\phi)\mathbf{k} \\
\mathbf{e}_\theta &= -\sin(\theta)\mathbf{i} + \cos(\theta)\mathbf{j} \\
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