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This can be proved by an argument similar to that used for the two-dimensional version. The physical interpretation is also similar: in a steady state, the rate of flow of particles escaping through S must balance the rate of creation of particles in E.

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As expected, this is the same as $\iiint_E \text{div}(\mathbf{u}) dV$.

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$E: \quad 0 \le r \le a, \quad -b \le z \le b, \quad 0 \le \theta \le 2\pi.$
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which is the same as $\iiint_E \text{div}(\mathbf{u})\, dV$, as expected.

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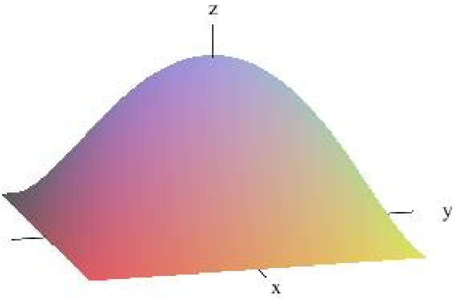

Divergence Theorem Example 3

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$$\left[x - \tfrac{1}{3}x^3 \right]_{x=-1}^{1} = (1 - \tfrac{1}{3}) - (-1 + \tfrac{1}{3}) = \tfrac{2}{3} - (-\tfrac{2}{3}) = \tfrac{4}{3} = \left[y - \tfrac{1}{3}y^3 \right]_{y=-1}^{1}$$

$$\iiint_E \operatorname{div}(\mathbf{u})\, dV = 2 \times \tfrac{4}{3} \times \tfrac{4}{3} = 32/9.$$

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$E: \quad -1 \leq x, y \leq 1, \quad 0 \leq z \leq f(x,y) = (1-x^2)(1-y^2)$
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Let E be the solid cube where $0 \leq x, y, z \leq \pi$.

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Let S be the surface of E, consisting of six faces:

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Let $\mathbf{u}$ be the vector field $(\cos(x),\ \cos(y),\ \cos(z))$. Then

$$\text{div}(\mathbf{u}) = \frac{\partial}{\partial x}(\cos(x)) + \frac{\partial}{\partial y}(\cos(y)) + \frac{\partial}{\partial z}(\cos(z)) = -\sin(x) - \sin(y) - \sin(z).$$

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$$\iiint_E \sin(x)\, dV = \int_{x=0}^{\pi} \int_{y=0}^{\pi} \int_{z=0}^{\pi} \sin(x)\, dx\, dy\, dz$$

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Similarly $\iiint_E -\sin(y)\, dV = \iiint_E -\sin(z)\, dV = -2\pi^2$.

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Similarly $\iiint_E -\sin(y)\, dV = \iiint_E -\sin(z)\, dV = -2\pi^2$. Altogether we have

$$\iiint_E \operatorname{div}(\mathbf{u})\, dV = -\iiint_E \sin(x)\, dV - \iiint_E \sin(y)\, dV - \iiint_E \sin(z)\, dV = -6\pi^2.$$

Divergence Theorem Example 4

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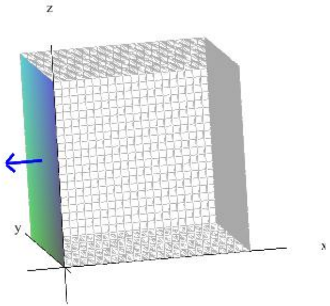

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Similarly, $\iint_{S_k} \mathbf{u}.\mathbf{n} dA = -\pi^2$ for all k.

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$$\iint_{S_3} \mathbf{u}.\mathbf{n}dA = \iint_{S_3} -1dA = -\text{area}(S_3) = -\pi^2.$$

Similarly, $\iint_{S_k} \mathbf{u}.\mathbf{n}dA = -\pi^2$ for all k. There are six faces so $\iint_S \mathbf{u}.\mathbf{n}dA = -6\pi^2$.