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On this curve we have

$$\begin{aligned} \mathbf{u} &= (-y,\ x,\ 0) = (-\sin(s),\ \cos(s),\ 0) \\ d\mathbf{r} &= (-\sin(s), \cos(s), -2\sin(2s))\,ds \\ \mathbf{u}.d\mathbf{r} &= (\sin^2(s) + \cos^2(s))\,ds = ds \end{aligned}$$

Stokes's Theorem Example 1

$S : (x, y, z) = (r\cos(s), r\sin(s), r^2\cos(2s))$,
$d\mathbf{A} = (-2r^2\cos(s),\ 2r^2\sin(s),\ r)\,dr\,ds$

Next, we have $\text{curl}(\mathbf{u}) = \det\begin{bmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ -y & x & 0 \end{bmatrix} = (0, 0, 2)$, so

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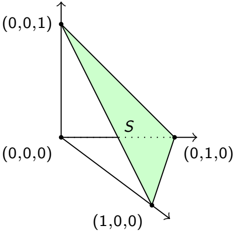
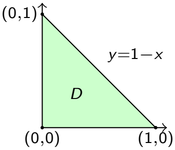
As expected, this is the same as $\iint_S \mathrm{curl}(\mathbf{u}).d\mathbf{A}$.

Let S be the triangular surface shown on the left below, given by $x + y + z = 1$ with $x, y, z \geq 0$.





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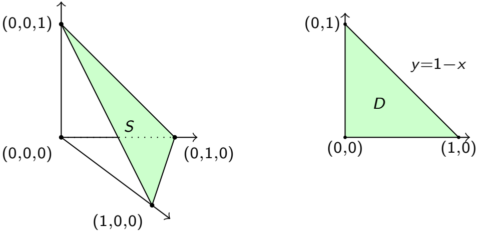

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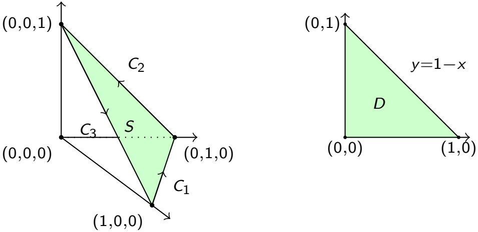

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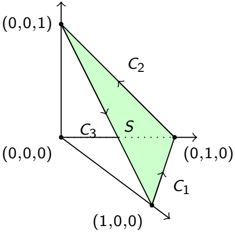

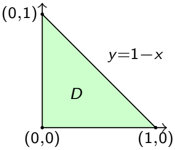

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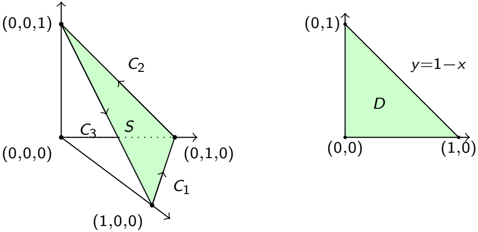

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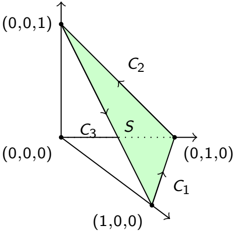

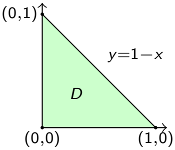

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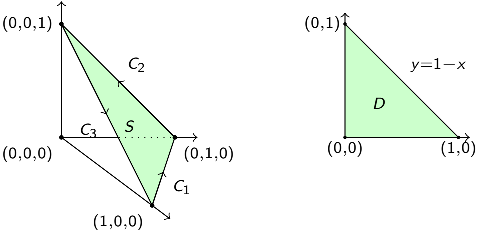

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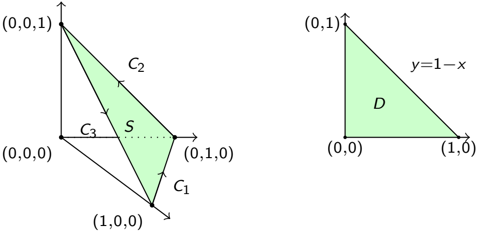

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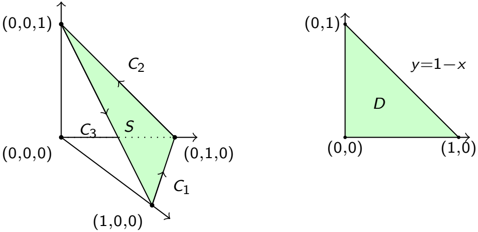

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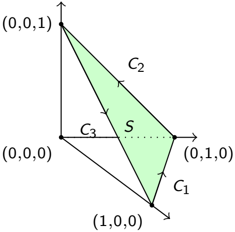

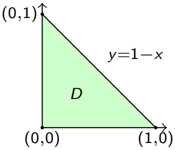

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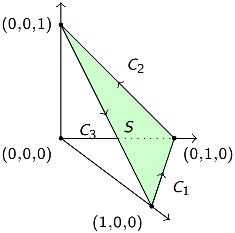

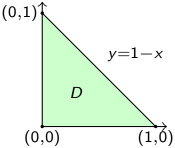

The other edges work in the same way, as in the following table:

edge	C_1	C_2	C_3
$\mathbf{r}$	$(1-t, t, 0)$	$(0, 1-t, t)$	$(t, 0, 1-t)$
$d\mathbf{r}$	$(-1, 1, 0)dt$	$(0, -1, 1)dt$	$(1, 0, -1)dt$
$\mathbf{u}$	$(0, 1-t, t)$	$(t, 0, 1-t)$	$(1-t, t, 0)$
$\mathbf{u}.d\mathbf{r}$	$(1-t)dt$	$(1-t)dt$	$(1-t)dt$
$\int \mathbf{u}.d\mathbf{r}$	$1/2$	$1/2$	$1/2$

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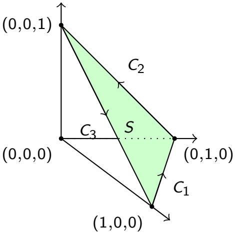

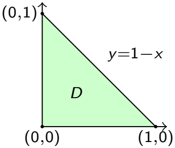

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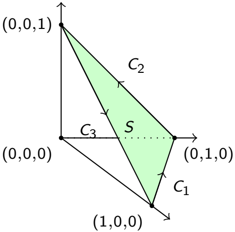
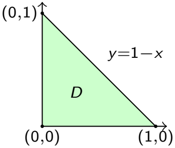
$\int \mathbf{u}.d\mathbf{r}$	$1/2$	$1/2$	$1/2$

Altogether, we have $\int_C \mathbf{u}.d\mathbf{r} = 3/2$.

Stokes's Theorem Example 2

$S : x + y + z = 1$ with $x, y, z \geq 0$; $\quad \mathbf{u} = (z, x, y)$; $\quad \int_C \mathbf{u}.d\mathbf{r} = 3/2.$





On the other hand, we have $\text{curl}(\mathbf{u}) = \det \begin{bmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ z & x & y \end{bmatrix}$

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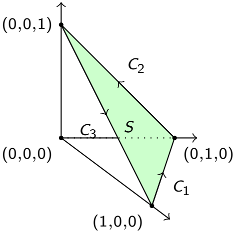

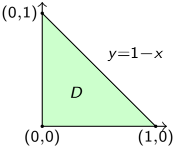

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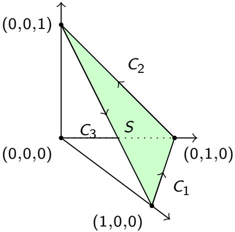

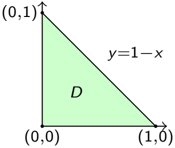

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The shadow of S in the xy-plane is the triangle D shown on the right.

Stokes's Theorem Example 2

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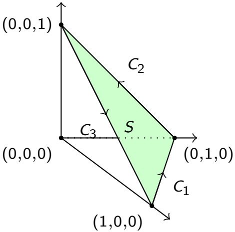

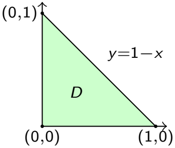

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Stokes's Theorem Example 2

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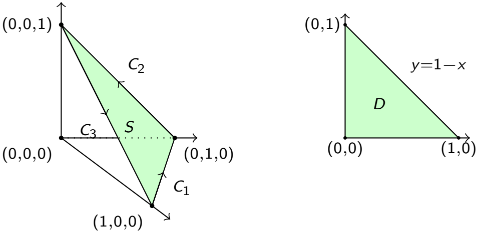

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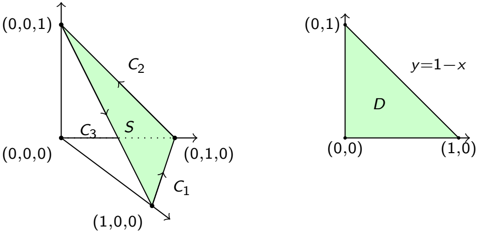

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Stokes's Theorem Example 2

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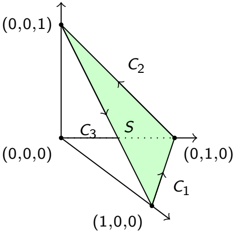

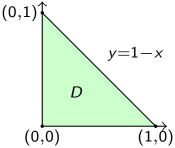

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The shadow of S in the xy-plane is the triangle D shown on the right. The surface has the form $z = f(x, y)$, where $f(x, y) = 1 - x - y$ and (x, y) lies in D, so $d\mathbf{A} = (-f_x, -f_y, 1)\, dx\, dy = (1, 1, 1)\, dx\, dy$. This gives

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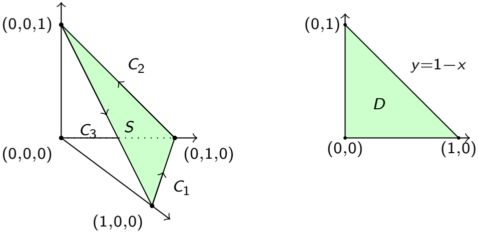
Stokes's Theorem Example 2

$S : x + y + z = 1$ with $x, y, z \geq 0$; $\quad \mathbf{u} = (z, x, y)$; $\quad \int_C \mathbf{u}.d\mathbf{r} = 3/2.$



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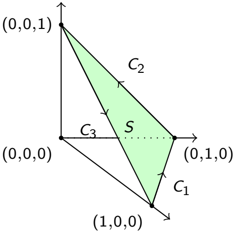

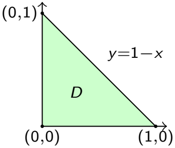

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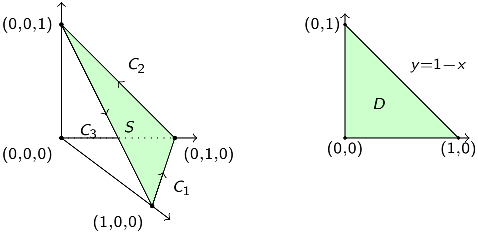

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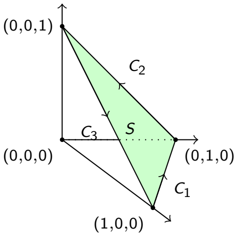

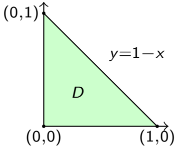

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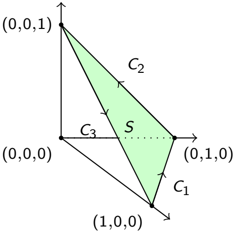

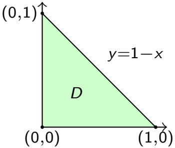

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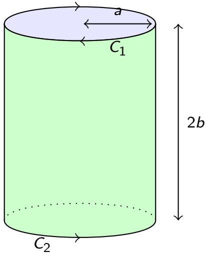

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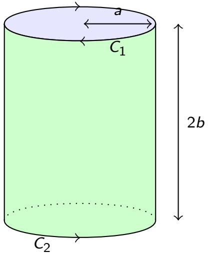

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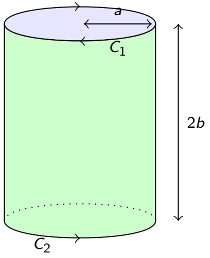

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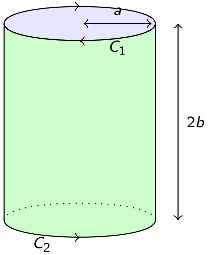

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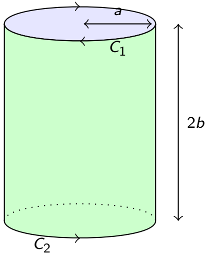

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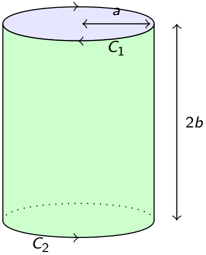

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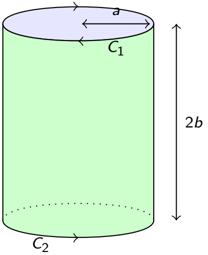

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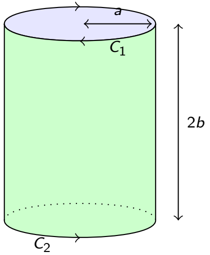

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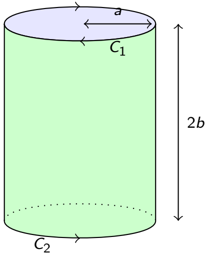

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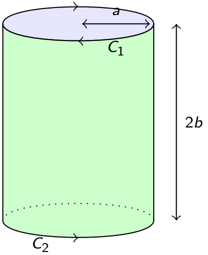

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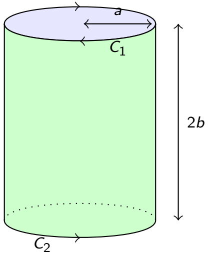

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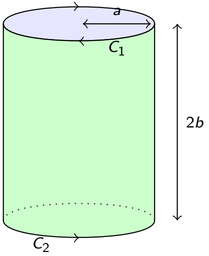

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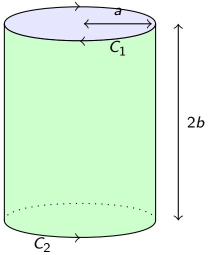

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On the surface S this becomes $\text{curl}(\mathbf{u}) = (-a\cos(\theta),\ -a\sin(\theta),\ 2z)$

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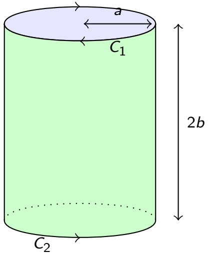

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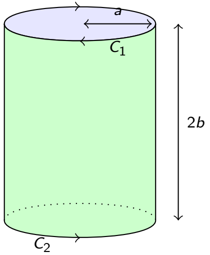

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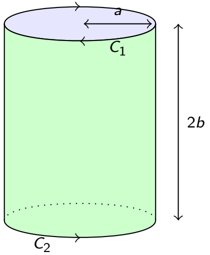

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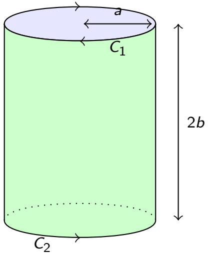

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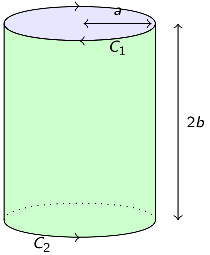

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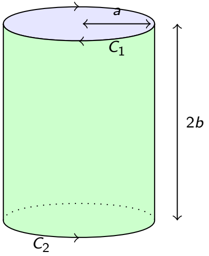

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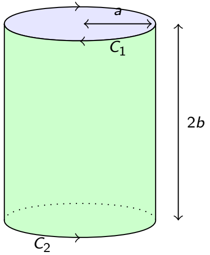

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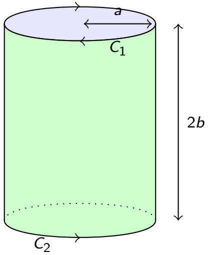

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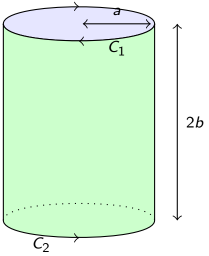

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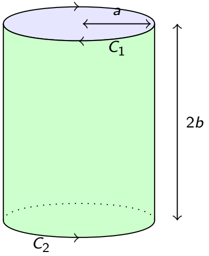

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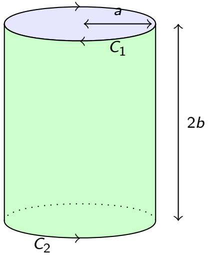

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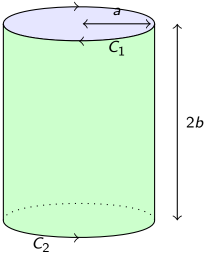

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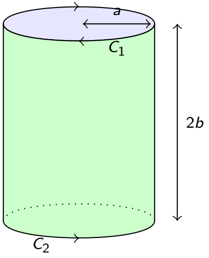

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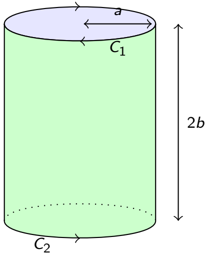

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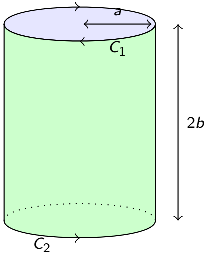

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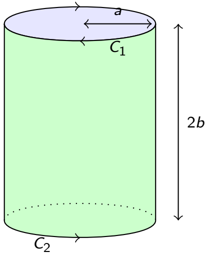

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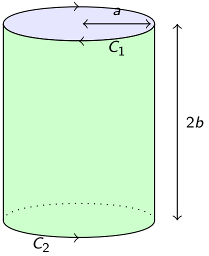

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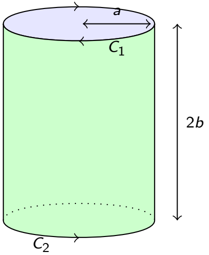

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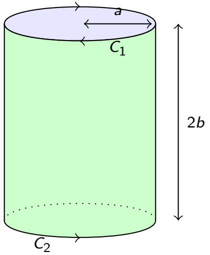

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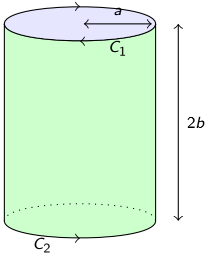
$$\int_{C_1} \mathbf{u}.d\mathbf{r} = \int_{t=0}^{2\pi} -a^2b\,dt = -2\pi a^2 b.$$

C_2 is similar: $d\mathbf{r} = (-a\sin(t), a\cos(t), 0)\,dt$, $\mathbf{u} = (ab\sin(t), -ab\cos(t), b^2)$, $\mathbf{u}.d\mathbf{r} = -a^2b\sin^2(t) - a^2b\cos^2(t) = -a^2b$

$S =$ cylinder: $r = a$ with $-b \leq z \leq b$ and $0 \leq \theta \leq 2\pi$. $\mathbf{u} = (-zy, zx, z^2)$. $\iint_S \text{curl}(\mathbf{u}).d\mathbf{A} = -4\pi a^2 b$.



Boundary of S: C_1 and C_2. Directions as shown keep S on the left when walking with head in the direction of $d\mathbf{A}$, away from the z-axis. Compatible parametrisations:

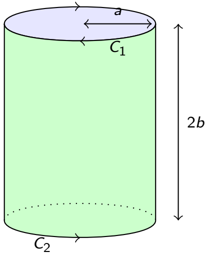
$$C_1: (x, y, z) = (a\cos(t), -a\sin(t), b)$$
$$C_2: (x, y, z) = (a\cos(t), a\sin(t), -b).$$

On C_1:

$$d\mathbf{r} = (-a\sin(t), -a\cos(t), 0)\,dt$$
$$\mathbf{u} = (-zy, zx, z^2) = (ab\sin(t), ab\cos(t), b^2)$$
$$\mathbf{u}.d\mathbf{r} = -a^2 b\sin^2(t) - a^2 b\cos^2(t) = -a^2 b$$
$$\int_{C_1} \mathbf{u}.d\mathbf{r} = \int_{t=0}^{2\pi} -a^2 b\,dt = -2\pi a^2 b.$$

C_2 is similar: $d\mathbf{r} = (-a\sin(t), a\cos(t), 0)\,dt$, $\mathbf{u} = (ab\sin(t), -ab\cos(t), b^2)$, $\mathbf{u}.d\mathbf{r} = -a^2 b\sin^2(t) - a^2 b\cos^2(t) = -a^2 b$, $\int_{C_2} \mathbf{u}.d\mathbf{r} = -2\pi a^2 b$.

S = cylinder: $r = a$ with $-b \leq z \leq b$ and $0 \leq \theta \leq 2\pi$. $\mathbf{u} = (-zy, zx, z^2)$. $\iint_S \text{curl}(\mathbf{u}).d\mathbf{A} = -4\pi a^2 b$.



Boundary of S: C_1 and C_2. Directions as shown keep S on the left when walking with head in the direction of $d\mathbf{A}$, away from the z-axis. Compatible parametrisations:

$$C_1 : (x, y, z) = (a\cos(t), -a\sin(t), b)$$
$$C_2 : (x, y, z) = (a\cos(t), a\sin(t), -b).$$

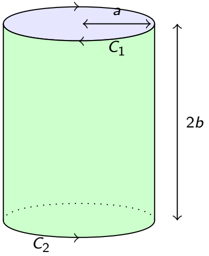
On C_1:

$$d\mathbf{r} = (-a\sin(t), -a\cos(t), 0)\,dt$$
$$\mathbf{u} = (-zy, zx, z^2) = (ab\sin(t), ab\cos(t), b^2)$$
$$\mathbf{u}.d\mathbf{r} = -a^2 b\sin^2(t) - a^2 b\cos^2(t) = -a^2 b$$
$$\int_{C_1} \mathbf{u}.d\mathbf{r} = \int_{t=0}^{2\pi} -a^2 b\,dt = -2\pi a^2 b.$$

C_2 is similar: $d\mathbf{r} = (-a\sin(t), a\cos(t), 0)\,dt$, $\mathbf{u} = (ab\sin(t), -ab\cos(t), b^2)$, $\mathbf{u}.d\mathbf{r} = -a^2 b\sin^2(t) - a^2 b\cos^2(t) = -a^2 b$, $\int_{C_2} \mathbf{u}.d\mathbf{r} = -2\pi a^2 b$. Putting these together, we get $\int_C \mathbf{u}.d\mathbf{r} = -4\pi a^2 b$

Stokes's Theorem Example 3

$S =$ cylinder: $r = a$ with $-b \leq z \leq b$ and $0 \leq \theta \leq 2\pi$. $\mathbf{u} = (-zy, zx, z^2)$. $\iint_S \mathrm{curl}(\mathbf{u}).d\mathbf{A} = -4\pi a^2 b$.



Boundary of S: C_1 and C_2. Directions as shown keep S on the left when walking with head in the direction of $d\mathbf{A}$, away from the z-axis. Compatible parametrisations:

$$C_1 : (x, y, z) = (a\cos(t), -a\sin(t), b)$$
$$C_2 : (x, y, z) = (a\cos(t), a\sin(t), -b).$$

On C_1:

$$d\mathbf{r} = (-a\sin(t), -a\cos(t), 0)\,dt$$
$$\mathbf{u} = (-zy, zx, z^2) = (ab\sin(t), ab\cos(t), b^2)$$
$$\mathbf{u}.d\mathbf{r} = -a^2b\sin^2(t) - a^2b\cos^2(t) = -a^2b$$
$$\int_{C_1} \mathbf{u}.d\mathbf{r} = \int_{t=0}^{2\pi} -a^2b\,dt = -2\pi a^2 b.$$

C_2 is similar: $d\mathbf{r} = (-a\sin(t), a\cos(t), 0)\,dt$, $\mathbf{u} = (ab\sin(t), -ab\cos(t), b^2)$, $\mathbf{u}.d\mathbf{r} = -a^2b\sin^2(t) - a^2b\cos^2(t) = -a^2b$, $\int_{C_2} \mathbf{u}.d\mathbf{r} = -2\pi a^2 b$. Putting these together, we get $\int_C \mathbf{u}.d\mathbf{r} = -4\pi a^2 b$, which is the same as $\iint_S \mathrm{curl}(\mathbf{u}).d\mathbf{A}$, as expected.