MAS243 PROBLEM BOOK

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1. Optimisation

Exercise 1. Find the maximum and minimum values of the function $f(x) = x^3 - 9x^2 + 15x + 35$ with $0 \leq x \leq 8$.

Exercise 2. Find and classify the critical points of the function $f(x) = \cos(x) - \cos^3(x)/3$. What is the unusual feature of this example?

Exercise 3. Show that $\sqrt{2}\sin(x - \pi/4) = \sin(x) - \cos(x)$. Using this, find the maximum value of $e^{-x}\sin(x)$ for $x \geq 0$.

Exercise 4. For the following functions, calculate the partial derivatives f_x and f_y, and verify that $f_{xy} = f_{yx}$.

(a) $f(x, y) = x^3 + 3x^2y + xy^2 + 4y^3$

(b) $f(x, y) = xy^2 \ln(x^2 + y^2)$.

Exercise 5. Suppose we have a function $f(x, y)$ of two variables. We say that f is biharmonic if it satisfies the equation

$$f_{xxxx} + 2f_{xxyy} + f_{yyyy} = 0.$$

(This comes up in the theory of small elastic deformations of nearly rigid bodies.) Show that the function $f(x, y) = xy^2(x^2 - y^2)$ is biharmonic but the function $g(x, y) = e^{x+y}$ is not.

Exercise 6. Show that the function $f(x, t) = t^{-1/2}e^{-x^2/t}$ satisfies the equation $4f_t = f_{xx}$. (This is relevant to the equations of heat flow, and also to the pricing of financial derivatives.)

Exercise 7. Find the critical points of the function $f(x, y) = (x + y + 2)e^{-(x^2+y^2)/2}$.

Exercise 8. Let ϕ be a constant. The function

$$f(x, y) = (x^2 + y^2)^2 - x\cos(\phi)/2 - y\sin(\phi)/2$$

has only one critical point. Find it.

2. Constrained optimisation

Exercise 9. Show that the function $u = x^3 + x^2y - y^2 - 2x^2$ has critical points at $(0, 0)$, $(1, 1/2)$ and $(-4, 8)$, and determine their nature.

Exercise 10. Find all the critical points of the function

$$u = x^3 - 3xy^2 + 4y^3 - 18y$$

and determine their nature.

Exercise 11. Find the maximum and minimum values of the function

$$f(x, y) = \frac{1}{1 + (x - 5)^2} + \frac{1}{1 + (y - 10)^2}.$$

Do this by thinking intelligently about the problem, not by grinding through the general method.

Exercise 12. Find and classify the critical points of the function

$$f(x, y, z) = 1 - x^2 - y^2 - z^2 + 2xyz.$$

Exercise 13. Locate the critical points of $f(x, y) = x^2y$ subject to the constraint $x^2 + xy = 1$.

Exercise 14. Locate the critical points of $f(x, y) = x^2+y^2$ subject to the condition $3x^2+4xy+6y^2 = 140$.

Exercise 15. A triangle in the xy-plane has vertices at $(0, 0)$, $(x, 0)$ and (x, y), with $x, y \geq 0$. The point (x, y) lies on the circle of radius 1 with centre at $(1, 0)$. Use the method of Lagrange multipliers to show that the maximum possible area for the triangle is $3\sqrt{3}/8$.

Exercise 16. A solid body of volume V and surface S is formed by joining together two cubes of different sizes so that every point of a face of the smaller cube is in contact with the larger cube. If $S = 7m^2$, use the method of Lagrange multipliers to find the critical value of V for which both cubes have non-zero volumes.

3. Integrals over plane regions

Exercise 17. Evaluate the following integrals, and sketch the corresponding regions in the (x, y)-plane.

(a) $\int_{x=-1}^{1} \int_{y=-2}^{2} (2x^2 + y^2)\, dy\, dx$
(b) $\int_{x=1}^{2} \int_{y=0}^{1} x\, e^y\, dy\, dx$
(c) $\int_{x=2/a}^{4/a} \int_{y=1/x}^{a} y^2\, dy\, dx$
(d) $\int_{y=0}^{\pi} \int_{x=0}^{\sin(y)} 1\, dx\, dy$.

Exercise 18. Express the following double integrals as repeated integrals and evaluate them:

(a) $\iint_D xy\, dA$, where D is the rectangle bounded by the lines $x = 0$, $x = a$, $y = 0$ and $y = b$.
(b) $\iint_D e^{x+y}\, dA$, where D is the region bounded by the lines $x = 0$, $y = 0$ and $x + y = 1$.
(c) $\iint_D e^{y^2}\, dA$, where D is the triangle with vertices $(0, 0)$, $(-1, 1)$ and $(1, 1)$.
(d) $\iint_D x^2\, dA$, where D is the trapezium with vertices $(-1, 0)$, $(1, 0)$, $(0, 1)$ and $(1, 1)$.

Exercise 19. By sketching the region of integration, show that

$$\int_{y=0}^{1} \int_{x=\sqrt{y}}^{1} 1\, dx\, dy = \int_{x=0}^{1} \int_{y=0}^{x^2} 1\, dy\, dx.$$

Evaluate both integrals and check that they are the same.

Exercise 20. Sketch the region of the (x, y)-plane over which the integral

$$I = \int_{x=0}^{1} \int_{y=1}^{x^2+1} f(x, y)\, dy\, dx$$

is taken. Obtain a similar expression for I by reversing the order of integration.

Exercise 21. Change the order of integration in the following integrals, and hence evaluate them:

(a) $\displaystyle\int_{y=0}^{\infty} \int_{x=y}^{\infty} \frac{e^{-x}}{x} dx\, dy$
(b) $\displaystyle\int_{x=0}^{a} \int_{y=x}^{a} \frac{y^2}{(x^2 + y^2)^{1/2}} dy\, dx$

You may find the following integral useful:

$$\int \frac{1}{(x^2 + y^2)^{1/2}} dx = \ln(x + \sqrt{x^2 + y^2}).$$

4. Plane polar integrals and volume integrals

Exercise 22. Consider the integral given in polar coordinates by $I = \int_{\theta=0}^{\pi/2} \int_{r=0}^{1} r^2 \sin(\theta)\, dr\, d\theta$. Sketch the corresponding region in the (x, y)-plane, and evaluate the integral.

Exercise 23. Evaluate $\displaystyle\iint_D xy\, dA$, where D is the quadrant of the disk $x^2 + y^2 \leq a^2$ where $x \geq 0$ and $y \geq 0$. (Hint: use polar coordinates.)

Exercise 24. Evaluate the following integrals, where D is the region given by $x^2 + y^2 \leq a^2$.

(a) $\displaystyle\iint_D (x^2 + y^2)^{\frac{1}{2}}\, dA$

(b) $\displaystyle\iint_D e^{-(x^2+y^2)}\, dA.$

Exercise 25. Evaluate $\iint_D x^2\, dA$, where D is the ring-shaped region given by $4 \leq x^2 + y^2 \leq 9$.

Exercise 26. Use polar coordinates to evaluate $\displaystyle\iint_D e^{-\sqrt{x^2+y^2}}\, dA$, where D is the region given by $x \geq 0$.

Exercise 27. Evaluate $\displaystyle\int_{x=0}^{1} \int_{y=0}^{1} \int_{z=0}^{1} xyz\, dz\, dy\, dx.$

Exercise 28. Evaluate $\displaystyle\int_{x=0}^{1} \int_{y=0}^{1} \int_{z=\sqrt{x^2+y^2}}^{2} xyz\, dz\, dy\, dx.$

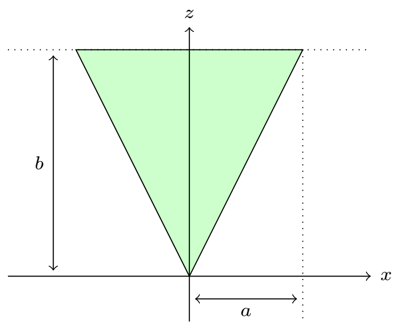
Exercise 29. Evaluate $\displaystyle\int_{x=0}^{1} \int_{y=x}^{1} \int_{z=y}^{1} x\, dz\, dy\, dx.$

Exercise 30. The region D in the (x, y)-plane is given by $|x| \leq 1$ and $|y| \leq 1$, and the surface S consists of the points (x, y, z) where (x, y) lies in D and $z = x^2 + xy$. Let E be the three-dimensional region between D and S. What is the volume of E?

5. Cylindrical and spherical integrals

Exercise 31. The point P has rectangular coordinates $x = y = 0$ and $z = 1$, and the point Q has cylindrical coordinates r, θ and z. What is the distance from P to Q?

Exercise 32. Let D be a flat-topped circular cone with cross-section as shown below:



Fint the centre of mass and the moment of inertia about the z-axis (assuming that the density is 1).

Exercise 33. Let D be the solid given by $x, y, z \geq 0$ and $x^2+y^2 \leq 1$ and $z \leq 2$. By rewriting everything in cylindrical coordinates, evaluate the integral

$$I = \iiint_D x + y + z\, dV.$$

Exercise 34. Let D be a solid cylinder, with base of radius a centred at the origin in the (x, y)-plane, and height h. The charge density on D is given by $\rho(x, y, z) = x^2 \sin(\pi z/h)$. Find the total charge.

Exercise 35. Let P be the shape obtained by joining together points $A, \dots, E$ with spherical coordinates as listed below:

	A	B	C	D	E
r	1	1	1	1	1
θ	$\pi/4$	$3\pi/4$	$5\pi/4$	$7\pi/4$	0
ϕ	$\pi/2$	$\pi/2$	$\pi/2$	$\pi/2$	0

Sketch P and describe it geometrically.

Exercise 36. Let Q be the surface given in spherical coordinates by the equation $\tan(\phi)\sin(\theta) = 1$. Explain why Q is a plane and give an equation for Q in terms of rectangular coordinates.

Exercise 37. Suppose that $b > a > 0$, and let D be the three-dimensional region between the spheres $x^2 + y^2 + z^2 = a^2$ and $x^2 + y^2 + z^2 = b^2$. Evaluate the integral

$$I = \iiint_D \frac{1}{(x^2 + y^2 + z^2)^{3/2}} dV.$$

Exercise 38. By rewriting everything in spherical polar coordinates, evaluate the integral

$$I = \int_{x=0}^{\infty} \int_{y=0}^{\infty} \int_{z=0}^{\infty} \exp\left(-(x^2 + y^2 + z^2)^{3/2}\right) dz\, dy\, dx.$$

6. Vector algebra and gradients

Unless otherwise specified, $\mathbf{r}$ refers to the position vector $\mathbf{r} = (x, y, z)$, and $r = |\mathbf{r}| = \sqrt{x^2 + y^2 + z^2}$.

Exercise 39. Consider the vectors $\mathbf{p} = 2\mathbf{i} - \mathbf{j}$, $\mathbf{q} = 4\mathbf{i} - 2\mathbf{j}$ and $\mathbf{r} = 2\mathbf{i} + 4\mathbf{j}$. Which of them are parallel to each other, and which of them are perpendicular to each other?

Exercise 40. If $\mathbf{a} = (2, -1, 2)$, $\mathbf{b} = (-1, 2, 1)$ and $\mathbf{c} = (1, -2, 1)$, find the following quantities:

(a) $|\mathbf{a}|$, $|\mathbf{b}|$ and $|\mathbf{c}|$.
(b) $\mathbf{a}.\mathbf{b}$, $\mathbf{a}.\mathbf{c}$ and $\mathbf{b}.\mathbf{c}$.
(c) $\mathbf{a} \times \mathbf{b}$, $\mathbf{a} \times \mathbf{c}$ and $\mathbf{b} \times \mathbf{c}$.
(d) The unit vector $\widehat{\mathbf{a}}$.
(e) The angle between $\mathbf{b}$ and $\mathbf{c}$.
(f) The area of the parallelogram spanned by $\mathbf{a}$ and $\mathbf{c}$.
(g) The component of $\mathbf{a}$ parallel to $\mathbf{b}$.
(h) The component of $\mathbf{a}$ perpendicular to $\mathbf{b}$.

Be sure to type-check your answers: do not give a vector where a scalar is required, or *vice-versa*.

Exercise 41. Consider vectors $\mathbf{a} = (u, v, w)$ and $\mathbf{b} = (x, y, z)$. Give formulae for $\mathbf{a}.\mathbf{b}$ and $\mathbf{a} \times \mathbf{b}$ and $|\mathbf{a} \times \mathbf{b}|^2$. Verify by direct expansion that

$$(\mathbf{a}.\mathbf{b})^2 + |\mathbf{a} \times \mathbf{b}|^2 = |\mathbf{a}|^2 |\mathbf{b}|^2.$$

Exercise 42. Find $\text{grad}(f)$ for the following functions

(a) $f = x^2 y + y^2 z + z^2 x$
(b) $f = \sin(r)/r$
(c) $f = e^{-x^2 - y^2} + z$.

Exercise 43. If $f = x^2 y z^3$ and $\mathbf{n} = (\frac{1}{3}, \frac{2}{3}, \frac{2}{3})$, find the directional derivative $\mathbf{n}.\nabla(f)$.

Exercise 44. The scalar field f is given by $f = x\sin(xy) + z\cos(xy)$. Find the component of $\text{grad}(f)$ in parallel to $(-1, 1, -1)$ at the point $(\pi/2, 2, 0)$.

Exercise 45. Put $f = x^2 - z^2$ and $g = 2xz + y^2$. Show that $\nabla(f)$ is always perpendicular to $\nabla(g)$.

7. Div and curl

Exercise 46. Find $\nabla.\mathbf{u}$ and $\nabla \times \mathbf{u}$ for the following vector fields:

$$\text{(a):} \qquad \mathbf{u} = (xy, yz, 0) \qquad\qquad \text{(b):} \qquad \mathbf{u} = (z, x, y).$$

Exercise 47. Consider a vector field of the form $\mathbf{u} = f(r)\mathbf{r}$, where f is a function of r only. Show that $\nabla.\mathbf{u} = 3\,f(r) + r\,f'(r)$. Show that if $\nabla.\mathbf{u} = 0$, then $f(r) = c/r^3$ for some constant c.
[**Hint:** remember the chain rule $\frac{\partial}{\partial x} f(r) = f'(r)\frac{\partial r}{\partial x}$.]

Exercise 48. Find constants a, b and c such that the vector field

$$\mathbf{v} = (x + 2y + az,\ bx - 3y - z,\ 4x + cy + 2z)$$

satisfies $\text{curl}(\mathbf{v}) = 0$. For these values of a, b and c, find a potential function f with $\text{grad}(f) = \mathbf{v}$.

Exercise 49. If $r = \sqrt{x^2 + y^2 + z^2}$, show that $\nabla^2(r^n) = n(n+1)r^{n-2}$.

Exercise 50. Let Ω be a scalar field, and let $\mathbf{F}$ be a vector field. Show that

(a) $\text{curl}(\Omega\mathbf{F}) = \Omega\ \text{curl}(\mathbf{F}) - \mathbf{F} \times \text{grad}(\Omega)$
(b) $\text{curl}(\text{grad}(\Omega)) = 0$.

Rewrite these identities in ∇ notation.

Exercise 51. Let $\mathbf{H}$ be a vector field that can be expressed as $\mathbf{H} = f\ \text{grad}(g)$ for some scalar fields f and g. Show that $\mathbf{H}$ is perpendicular to $\text{curl}(\mathbf{H})$ at every point.
[**Hint:** use the previous question.]

Now consider the vector field $\mathbf{H} = x^2y\,\mathbf{r}$ (where $\mathbf{r} = (x, y, z)$ as usual). Find scalar fields f and g such that $\mathbf{H} = f\ \text{grad}(g)$. Calculate $\text{curl}(\mathbf{H})$ and check directly that it is perpendicular to $\mathbf{H}$.

Exercise 52. For the vector field $\mathbf{A} = (x^2y,\ y^2z,\ z^2x)$, calculate

(a) $\nabla.\mathbf{A}$
(b) $\nabla(\nabla.\mathbf{A})$
(c) $\nabla \times \mathbf{A}$
(d) $\nabla \times (\nabla \times \mathbf{A})$
(e) $\nabla^2(\mathbf{A})$.

Verify the identity $\nabla^2(\mathbf{A}) = \nabla(\nabla.\mathbf{A}) - \nabla \times (\nabla \times \mathbf{A})$ in this case.

Exercise 53. Show that for any vector fields $\mathbf{u} = (p,\ q,\ r)$ and $\mathbf{v} = (f,\ g,\ h)$ we have

$$\nabla.(\mathbf{u} \times \mathbf{v}) = (\nabla \times \mathbf{u}).\mathbf{v} - (\nabla \times \mathbf{v}).\mathbf{u}.$$

8. Polar fields and line integrals

Exercise 54. Let $\mathbf{u}$ be the vector field given in spherical polar coordinates by

$$\mathbf{u} = r^2 \cos(\phi)\mathbf{e}_r + r^{-1}\mathbf{e}_\phi + (r\,\sin(\phi))^{-1}\mathbf{e}_\theta$$

Find $\mathrm{div}(\mathbf{u})$ and $\mathrm{curl}(\mathbf{u})$.

Exercise 55. Let $\mathbf{u}$ be the vector field given in cylindrical polar coordinates by

$$\mathbf{u} = r\cos(\theta)\mathbf{e}_r + r\sin(\theta)\mathbf{e}_\theta + \mathbf{e}_z.$$

Find $\mathrm{div}(\mathbf{u})$ and $\mathrm{curl}(\mathbf{u})$.

Exercise 56. Consider the vector field $\mathbf{u} = r^{-2}\mathbf{r}$, where $\mathbf{r} = (x, y, z)$ and $r = |\mathbf{r}|$. Show that $\mathrm{curl}(\mathbf{u}) = 0$ and $\mathrm{div}(\mathbf{u}) = r^{-2}$.

Exercise 57. Let r denote $\sqrt{x^2 + y^2}$ as in cylindrical polar coordinates. Use the formula for ∇^2 in those coordinates to show that $\nabla^2(r^2 + z^2) = 6$.

Exercise 58. Evaluate $\int_C x^2\,|d\mathbf{r}|$, where C is the circle of radius a centred at the origin.

Exercise 59. Consider the vector field $\mathbf{u} = (-z, 0, x)$ and the following three paths from $(0, 0, 0)$ to $(1, 1, 1)$.

- C_1 is just a straight line.
- C_2 is given by $(x, y, z) = (t, t^2, t^3)$ for $0 \leq t \leq 1$.
- C_3 is given by $(x, y, z) = (\sin(\theta), 2\theta/\pi, 1 - \cos(\theta))$ for $0 \leq \theta \leq \pi/2$.

Write $I_1 = \int_{C_1} \mathbf{u}.d\mathbf{r}$ and $I_2 = \int_{C_2} \mathbf{u}.d\mathbf{r}$ and $I_3 = \int_{C_3} \mathbf{u}.d\mathbf{r}$.

(a) Would you expect I_1, I_2 and I_3 to be the same? Why?
(b) Calculate I_1, I_2 and I_3, and check your answer to (a).

Exercise 60. Let C be the curve given by

$$\mathbf{r} = (2^t\cos(10\pi t^2), 2^t\sin(10\pi t^2), 2\pi)$$

for $0 \leq t \leq 1$, and let $\mathbf{u}$ be the vector field

$$\mathbf{u} = (e^x\cos(y)\cos(z),\ -e^x\sin(y)\cos(z),\ -e^x\cos(y)\sin(z)).$$

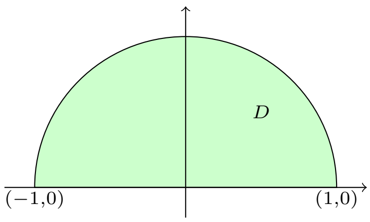
Calculate $\int_C \mathbf{u}.d\mathbf{r}$. Think carefully about the most efficient method before launching into calculations.

9. The two-dimensional divergence theorem and Green's theorem

Exercise 61. Let D be the disc of radius a centred at $(0,0)$, and let $\mathbf{u}$ be the vector field $(xy^2,0)$. Let C be the boundary curve of D. Verify the divergence theorem $\iint_D \operatorname{div}(\mathbf{u})\,dA = \int_C \mathbf{u}.d\mathbf{n}$ in this case.

Exercise 62. Show that $\cos^2(\theta)\sin(\theta) = (\sin(3\theta)+\sin(\theta))/4$.

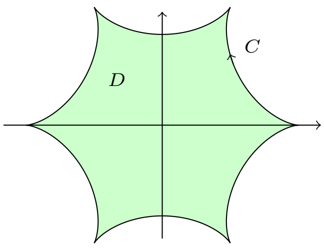
Exercise 63. Consider the region D as shown, and the vector field $\mathbf{u} = (0, x^4 + x^2y^2 - x^2)$.



Check the divergence theorem in this case. (The previous exercise will be helpful.)

Exercise 64. The following picture shows a hypocycloid curve C, which can be parametrised as

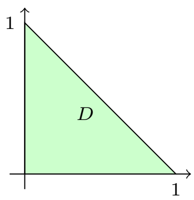
$$(x,y) = (5\cos(t)+\cos(5t),\ 5\sin(t)-\sin(5t)).$$



Use the divergence theorem to find the area of the region D enclosed by C. It may help to recall the identity

$$\cos(\alpha)\cos(\beta) = \tfrac{1}{2}\cos(\beta+\alpha) + \tfrac{1}{2}\cos(\beta-\alpha).$$

Exercise 65. Consider the following region D, and the vector field $\mathbf{u} = (-y^2, x^2)$. Verify Green's theorem for this case.



Exercise 66. Let f be any well-behaved function of two variables, and let C be the curve where $f(x,y)=1$. Suppose that this is a finite, closed curve like a circle, not branching or extending to infinity. Explain three different reasons why $\int_C \operatorname{grad}(f).d\mathbf{r} = 0$.

Exercise 67. Show that

$$\sin(\alpha)\cos^2(\alpha) = \tfrac{1}{4}(\sin(3\alpha) + \sin(\alpha))$$
$$\cos^3(\alpha) = \tfrac{1}{4}\cos(3\alpha) + \tfrac{3}{4}\cos(\alpha).$$

(You will need these identities in the questions below.)

Exercise 68. Evaluate $\iint_S \mathbf{F}.d\mathbf{A}$, where $\mathbf{F} = x^2\mathbf{i} + y^3\mathbf{j} + z^4\mathbf{k}$ and S is the surface of the cube bounded by the planes $x = 0$, $x = 1$, $y = 0$, $y = 1$, $z = 0$ and $z = 1$.

Exercise 69. Evaluate $\iint_S z^2\, dA$, where S is the hemispherical shell given by $x^2 + y^2 + z^2 = a^2$ with $z \geq 0$.

Exercise 70. Let S be the hemispherical shell given by $x^2 + y^2 + z^2 = 1$ with $x \geq 0$ (not $z \geq 0$). Evaluate $\iint_S \mathbf{F}.d\mathbf{A}$, where $\mathbf{F} = (1, -z, y)$.

Exercise 71.

(a) Let S be the cylindrical surface given parametrically by $x = a\cos(\theta)$ and $y = a\sin(\theta)$ with $0 \leq \theta \leq 2\pi$ and $0 \leq z \leq b$. Evaluate $\iint_S \mathbf{F}.d\mathbf{A}$, where $\mathbf{F} = (x^2, y, 0)$.

(b) Now let E be the solid region bounded by S together with the planes $z = 0$ and $z = b$. Evaluate $\iiint_E \operatorname{div}(\mathbf{F})dV$.

Exercise 72. Use the Divergence Theorem to evaluate $\iint_S \mathbf{F}.d\mathbf{A}$, where $\mathbf{F} = (y^2x, z^2y, x^2z)$ and S is the surface of the sphere of radius a centred at the origin.

Exercise 73. Let E be the solid cylinder with equations $0 \leq x^2 + y^2 \leq a^2$ and $0 \leq z \leq b$. Let S be the surface of E, and let $\mathbf{F}$ be the vector field (x^3, y^3, z^3). Use the Divergence Theorem to evaluate $\iint_S \mathbf{F}.d\mathbf{A}$.

Exercise 74. Let E be the cone whose top is a flat disc of radius a centred on the z-axis at height b, and whose point is at the origin. Let S_1 be the flat top of E, and let S_2 be the lower curved surface (so S_1 and S_2 together form the whole boundary of E).

(a) Give equations for S_1, S_2 and E in cylindrical polar coordinates.

(b) Put $\mathbf{F} = \operatorname{grad}(f)$, where $f = x^2 + y^2 + z^2$. Show that $\int_{S_2} \mathbf{F}.d\mathbf{A} = 0$, and calculate $\int_{S_1} \mathbf{F}.d\mathbf{A}$.

(c) Use the Divergence Theorem to deduce the volume of E.

Exercise 75. Let C be the vertical circle given by $y = a\sin(t)$ and $z = a\cos(t)$ with $x = 0$. Use Stokes's Theorem to evaluate $\int_C (x^2y, z, 0).d\mathbf{r}$. Check your answer by calculating the integral directly.

Exercise 76. Consider points

$$P = (0, 0, c) \qquad Q = (a, 0, c) \qquad R = (a, b, c).$$

Let C be the triangular path that goes from P to Q to R and back to P. Use Stokes's Theorem to evaluate $\int_C (yz^2, x^3, xy^2).d\mathbf{r}$.