MAS243 - examinable formulae

Formula sheet

You will get the following formula sheet in the exam:

$$\cos(A \pm B) = \cos A \cos B \mp \sin A \sin B$$
$$\sin(A \pm B) = \sin A \cos B \pm \cos A \sin B$$
$$\tan(A \pm B) = \frac{\tan A \pm \tan B}{1 \mp \tan A \tan B}$$
$$a\cos\theta + b\sin\theta = R\cos(\theta - \alpha) \text{ where } R = \sqrt{a^2+b^2} \text{ and } \cos\alpha = \tfrac{a}{R},\ \sin\alpha = \tfrac{b}{R}$$
$$\cos^2\theta = \frac{1}{2}(\cos 2\theta + 1)$$
$$\cos^3\theta = \frac{1}{4}(3\cos\theta + \cos 3\theta)$$
$$\cos^4\theta = \frac{1}{8}(3 + 4\cos 2\theta + \cos 4\theta)$$
$$\sin^2\theta = \frac{1}{2}(1 - \cos 2\theta)$$
$$\sin^3\theta = \frac{1}{4}(3\sin\theta - \sin 3\theta)$$
$$\sin^4\theta = \frac{1}{8}(3 - 4\cos 2\theta + \cos 4\theta)$$

Formulae that you need to know

Two-dimensional polar coordinates.

$$x = r\cos(\theta) \qquad y = r\sin(\theta)$$
$$r = \sqrt{x^2+y^2} \qquad \theta = \arctan(y/x)$$
$$\mathbf{e}_r = (\cos(\theta),\ \sin(\theta)) \qquad \mathbf{e}_\theta = (-\sin(\theta),\ \cos(\theta))$$
$$dA = r\,dr\,d\theta$$

Cylindrical polar coordinates.

$$x = r\cos(\theta) \qquad y = r\sin(\theta)$$
$$r = \sqrt{x^2+y^2} \qquad \theta = \arctan(y/x)$$
$$\mathbf{e}_r = (\cos(\theta),\ \sin(\theta),\ 0) \qquad \mathbf{e}_\theta = (-\sin(\theta),\ \cos(\theta),\ 0) \qquad \mathbf{e}_z = (0,0,1) = \mathbf{k}$$
$$dV = r\,dr\,d\theta\,dz$$

Spherical polar coordinates.

$$x = r\cos(\theta)\sin(\phi) \qquad r = \sqrt{x^2+y^2+z^2}$$
$$y = r\sin(\theta)\sin(\phi) \qquad \mathbf{e}_r = (\cos(\theta)\sin(\phi),\ \sin(\theta)\sin(\phi),\ \cos(\phi))$$
$$z = r\cos(\phi) \qquad dV = r^2\sin(\phi)\,dr\,d\theta\,d\phi$$

Vector algebra. For vectors $\mathbf{a} = (x,y,z)$ and $\mathbf{b} = (u,v,w)$ we have

$$\mathbf{a}.\mathbf{b} = xu + yv + zw \qquad \text{(a scalar)}$$

$$\mathbf{a}\times\mathbf{b} = (yw - zv,\ zu - xw,\ xv - yu) = \det\begin{bmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ x & y & z \\ u & v & w \end{bmatrix} \qquad \text{(a vector)}$$

$$\|\mathbf{a}\| = \sqrt{x^2+y^2+z^2} = \sqrt{\mathbf{a}.\mathbf{a}} \qquad \text{(a scalar)}$$

$$\widehat{\mathbf{a}} = \frac{\mathbf{a}}{\|\mathbf{a}\|} = \left(\frac{x}{\sqrt{x^2+y^2+z^2}},\ \frac{y}{\sqrt{x^2+y^2+z^2}},\ \frac{z}{\sqrt{x^2+y^2+z^2}}\right) \qquad \text{(a vector).}$$

Now let $\mathbf{n}$ be another vector, and let $\mathbf{a}_{||}$ and $\mathbf{a}_{\perp}$ be the components of $\mathbf{a}$ parallel and perpendicular to $\mathbf{n}$. If $\mathbf{n}$ is a unit vector (ie $\|\mathbf{n}\| = 1$) then

$$\mathbf{a}_{||} = (\mathbf{a}.\mathbf{n})\mathbf{n} \qquad \mathbf{a}_{\perp} = \mathbf{a} - \mathbf{a}_{||} = \mathbf{a} - (\mathbf{a}.\mathbf{n})\mathbf{n}.$$

If $\mathbf{n}$ is not necessarily a unit vector, we have

$$\mathbf{a}_{||} = \frac{\mathbf{a}.\mathbf{n}}{\mathbf{n}.\mathbf{n}}\mathbf{n} = (\mathbf{a}.\widehat{\mathbf{n}})\widehat{\mathbf{n}} \qquad \mathbf{a}_{\perp} = \mathbf{a} - \mathbf{a}_{||} = \mathbf{a} - \frac{\mathbf{a}.\mathbf{n}}{\mathbf{n}.\mathbf{n}}\mathbf{n}.$$

Differential operators. For a scalar field f and a vector field $\mathbf{u} = (p, q, r)$ we have

$$\nabla(f) = \mathrm{grad}(f) = (f_x, f_y, f_z) \qquad \text{(a vector field)}$$

$$\nabla^2(f) = f_{xx} + f_{yy} + f_{zz} \qquad \text{(a scalar field)}$$

$$\nabla.\mathbf{u} = \mathrm{div}(\mathbf{u}) = p_x + q_y + r_z \qquad \text{(a scalar field)}$$

$$\nabla\times\mathbf{u} = \mathrm{curl}(\mathbf{u}) = (r_y - q_z,\ p_z - r_x,\ q_x - p_y) = \det\begin{bmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ p & q & r \end{bmatrix} \qquad \text{(a vector field)}$$

$$\begin{aligned} \nabla^2(\mathbf{u}) &= (\nabla^2(p),\ \nabla^2(q),\ \nabla^2(r)) \\ &= (p_{xx} + p_{yy} + p_{zz},\ q_{xx} + q_{yy} + q_{zz},\ r_{xx} + r_{yy} + r_{zz}) \qquad \text{(a vector field)} \end{aligned}$$

Geometry of surfaces. You will need to be able to find the area element dA and the analogous vector quantity $d\mathbf{A}$ for a variety of different surfaces. In some cases it is easiest to find dA and the unit normal vector $\mathbf{n}$ separately and then $d\mathbf{A} = \mathbf{n}\,dA$. In other cases it is easiest to find $d\mathbf{A}$ first and then $dA = \|d\mathbf{A}\|$ and $\mathbf{n} = d\mathbf{A}/\|d\mathbf{A}\|$. In some cases the formulae below will give the inward normal when we want the outward normal, in which case we just need to multiply by minus one.

(a) For a surface where the position vector $\mathbf{r} = (x, y, z)$ is given in terms of two parameters u and v, we have

$$d\mathbf{A} = \mathbf{r}_u \times \mathbf{r}_v\ du\,dv = (x_u, y_u, z_u) \times (x_v, y_v, z_v)\ du\,dv.$$

(b) For a surface given in the form $z = f(x, y)$ we have

$$d\mathbf{A} = (-f_x, -f_y, 1)\,dx\,dy.$$

(c) For the curved surface of a cylinder of radius a centred on the z-axis, we have

$$\begin{aligned} \mathbf{n} &= (\cos(\theta), \sin(\theta), 0) \\ dA &= a\,dr\,dz \end{aligned}$$

(d) For a sphere of radius a centred at the origin, we have

$$\begin{aligned} \mathbf{n} &= (\cos(\theta)\sin(\phi),\ \sin(\theta)\sin(\phi),\ \cos(\phi)) \\ dA &= a^2\sin(\phi)d\theta\,d\phi \end{aligned}$$

Other formulae

If you need any other formulae in the exam, then the exam question will state the relevant formulae.