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Example 1:

There are 100 choices for who gets the gold. When we have awarded the gold, there are 99 choices left for who gets silver, and then 98 for who gets bronze. So the total number of ways in which the medals can be awarded is $100 \times 99 \times 98 = 970200 \simeq 9.7 \times 10^5$.

Example 2:

Similarly, there are $100 \times 99 \times 98$ possible choices for the list of people who are interviewed, in the order in which they are interviewed. But we do not care about the order, we only care about the set of interviewees. So we need to divide by the number of possible orders, which is $3! = 6$. Thus the number of ways to choose a set of three interviewees is
$100 \times 99 \times 98 / 6 = 161700 \simeq 1.6 \times 10^5$.

Example 4:
In the National Lottery, six balls are drawn from a set of 59 balls. So the number of possible outcomes is $\binom{59}{6} = \frac{59 \times 58 \times 57 \times 56 \times 55 \times 54}{6!}$
$= 45057474 \simeq 4.5 \times 10^7$.

Theorem 6:
The expansion of $(1+x)^n$ works as follows: for each subset $I \subseteq \{1, \dots, n\}$ we can take x from the i'th factor whenever $i \in I$, and take 1 from the i'th factor whenever $i \notin I$. If $|I| = k$ then we have taken k copies of x & $(n-k)$ copies of 1 so the product is x^k. Thus, for each set $I \subseteq \{1, \dots, n\}$ with $|I| = k$ we get a term x^k. This means that the coefficient of x^k is the number of such subsets I, which is $\binom{n}{k}$. So we get $(1+x)^n = \sum_k \binom{n}{k} x^k$. $\square$

Proposition 7:

Let P be the ~~number~~ set of subsets $A \subseteq \{1, \dots, n\}$ with $|A| = k$, so $|P| = \binom{n}{k}$. Let Q be the subset of those A for which $n \in A$, and let R be the subset of those A for which $n \notin A$. As $P = Q \cup R$ with no overlap, we have $\binom{n}{k} = |P| = |Q| + |R|$. Now $n \notin A$ iff A is really a subset of $\{1, \dots, n-1\}$, so $|R|$ is just the number of subsets of $\{1, \dots, n-1\}$ of size k, or in other words $|R| = \binom{n-1}{k}$.

On the other hand, the sets $A \in Q$ just have the form $A = B \cup \{n\}$ where $B \subseteq \{1, \dots, n-1\}$ with $|B| = k-1$. So $|Q|$ is the same as the number of such subsets B, or in other words $|Q| = \binom{n-1}{k-1}$

The equation $|P| = |Q| + |R|$ now gives $\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$ as claimed.

Alternatively, we can give an algebraic proof: we start with

$$\binom{n}{k} = \frac{n!}{k!\,(n-k)!}$$

On the top we have $n! = n \times (n-1) \times (n-2) \dots \times 1 = n \times (n-1)!$.
We can write the first factor as $k + (n-k)$. This gives

$$\binom{n}{k} = k \frac{(n-1)!}{k!\,(n-k)!} + (n-k) \frac{(n-1)!}{k!\,(n-k)!}$$

In the first term, we can rewrite $k!$ as $k \times (k-1)!$ & cancel k's to get $\frac{(n-1)!}{(k-1)!\,(n-k)!}$

In the 2nd term, we can rewrite $(n-k)!$ as $(n-k) \times (n-k-1)!$ & cancel to get $\frac{(n-1)!}{k!\,(n-k-1)!}$

Now $\binom{n-1}{k-1} = \frac{(n-1)!}{(k-1)!\,((n-1)-(k-1))!}$ & $(n-1)-(k-1) = n-k$ so $\binom{n-1}{k-1} = \frac{(n-1)!}{(k-1)!\,(n-k)!}$

Also $\binom{n-1}{k} = \frac{(n-1)!}{k!\,(n-k-1)!}$.

Putting this together, we get $\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$ as claimed.

Proposition 8

Suppose that $0 \leq k \leq n$. Let P be the collection of subsets $A \subseteq \{1, \dots, n\}$ with $|A| = k$, & let Q be the collection of subsets $B \subseteq \{1, \dots, n\}$ with $|B| = n-k$. Then $|P| = \binom{n}{k}$ & $|Q| = \binom{n}{n-k}$.

There is a one-to-one correspondence between A's & B's given by $A \mapsto A^c = \{i : i \notin A\}$ & $B \mapsto B^c = \{i : i \notin B\}$.
Thus the number of possible A's is the same as the number of possible B's, ie $\binom{n}{k} = \binom{n}{n-k}$. as claimed

Alternatively, we can give an algebraic proof:

$$\binom{n}{n-k} = \frac{n!}{(n-k)!\,(n-(n-k))!} = \frac{n!}{(n-k)!\,k!} = \binom{n}{k}. \quad \square$$

Example 9:
Consider an $n \times m$ grid. Any minimal path from the bottom left to the top right must take $n+m$ steps, of which n must be horizontal & m must be vertical. The route is freely & completely specified by saying which of the steps are horizontal.
Thus, the number of possible routes is the number of subsets of size n in $\{1, \dots, n+m\}$, or in other words $\binom{n+m}{n}$

The picture in the notes shows $n=4$, $m=3$ so $\binom{n+m}{n} = \binom{7}{4} = 35$.

Example 10:
Consider routes from the bottom left to top right of a 6×3 grid. The number of possible routes is $\binom{9}{6} = 84$ by Example 9.

Each route consists of four horizontal sections, of lengths x_1, x_2, x_3, x_4, separated by vertical steps. (If we take two vertical steps directly after each other, then that counts as a horizontal section of length 0 in between, so one of the x_i's will be zero. Similarly, if the first step is vertical, then that counts as an initial horizontal section of length 0, so $x_1 = 0$. In the same way, if the last step is vertical then $x_4 = 0$.)

As we take 6 horizontal steps altogether, we must have $x_1 + x_2 + x_3 + x_4 = 6$.

Conversely, if we are given a list $x_1, \ldots, x_4$ of nonnegative integers with $x_1 + \cdots + x_4 = 6$, then we can make a corresponding grid route by taking x_1 horizontal steps, one vertical step, x_2 horizontal steps, one vertical step, x_3 horizontal steps, one vertical step, and finally x_4 horizontal steps.

This gives a one-to-one correspondence between routes across the grid & nonnegative integer solutions to $x_1 + \cdots + x_4 = 6$, so the number of solutions is $\binom{9}{6} = 84$ again.

Proposition 11:
The proof is the obvious generalisation of the argument above, by considering routes across an $n \times (k-1)$ grid, and noting that each such route consists of horizontal segments of lengths $x_1, \ldots, x_k$ where $x_1 + \cdots + x_k = n$. $\square$

Example 12 / Proposition 13 :

Suppose we want to solve $y_1 + \dots + y_k = n$ with $y_1, \dots, y_k \geq 1$.
We can rewrite this in terms of the variables $x_i = y_i - 1$:
the conditions are $x_1 + \dots + x_k = (y_1 - 1) + \dots + (y_k - 1) = (y_1 + \dots + y_k) - k = n - k$
& $x_i \geq 0$.

Proposition 11 tells us that the number of solutions for
$x_1 + \dots + x_k = p$ (with $x_i \geq 0$) is $\binom{p+k-1}{k-1}$.
Taking $p = n-k$, we see that the number of solutions for
$x_1 + \dots + x_k = n-k$ is $\binom{n-1}{k-1}$.
By the transformation $y_i = x_i + 1$, these correspond to the solutions of $y_1 + \dots + y_k = n$ with $y_i \geq 1$.
Thus, the number of such solutions is $\binom{n-1}{k-1}$ □.

Example 14 :
Let x_i be the number of empty seats directly to the left of patient i (for $1 \leq i \leq k$) & let x_{k+1} be the number of empty seats to the right of the last patient.
Then $x_1 + \dots + x_{k+1}$ = total number of empty seats $= n-k$.
To ensure that no two patients are adjacent, we need
$x_2, \dots, x_k > 0$, but x_1 & x_{k+1} can be zero.
Put $w_1 = x_1$, $w_2 = x_2 - 1$, ..., $w_k = x_k - 1$, $w_{k+1} = x_{k+1}$
so $w_i \geq 0$ for all i &

$$w_1 + \dots + w_{k+1} = x_1 + \underbrace{(x_2 - 1) + \dots + (x_k - 1)}_{k-1 \text{ terms}} + x_{k+1}$$

$$= (x_1 + \dots + x_{k+1}) - (k-1) = n - k - (k-1) = n - 2k + 1.$$

By Proposition 11, the number of solutions is $\binom{(n-2k+1) + (k+1-1)}{k+1-1} = \binom{n-k+1}{k}$.

(In general: number of solutions $= \binom{(\text{total of variables}) + (\text{number of variables}) - 1}{\text{number of variables} - 1}$

Example 15 :

Example 14 can be restated more abstractly as follows :
The number of subsets $A \subseteq \{1,\dots,n\}$ with $|A| = k$ & no adjacent entries is $\binom{n-k+1}{k}$. For the lottery problem we have $n = 59$ & $k = 6$ so the number of possible lottery outcomes with no adjacent balls is $\binom{54}{6} = 25827165 \simeq 2.6 \times 10^7$.
The total number of lottery outcomes is $\binom{59}{6} \simeq 4.5 \times 10^7$, so the number with at least one adjacent pair is $\binom{59}{6} - \binom{54}{6} \simeq 1.9 \times 10^7$

The proportion with at least one adjacent pair is $\dfrac{\binom{59}{6} - \binom{54}{6}}{\binom{59}{6}} \simeq 0.42$.

Example 16 :

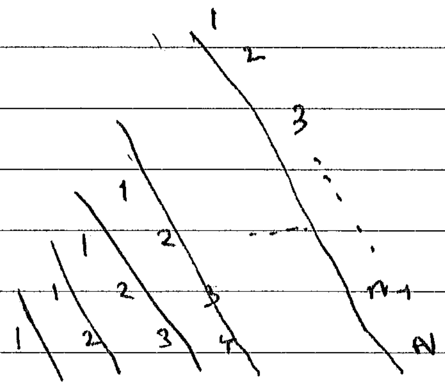
Let P be the collection of subsets $A \subseteq \{1,\dots,n\}$ with $|A| = k$, so $|P| = \binom{n}{k}$. If $A \in P$ then the maximum element of A must be less than or equal to n. It is also easy to see that the maximum must be at least k (which occurs only in the case $A = \{1,\dots,k\}$). Thus, if we put
$P_m = \{A \in P : \max(A) = m\}$, we find that
$P = P_k \cup P_{k+1} \cup \dots \cup P_n$, & these sets do not overlap.
Thus, we have $\binom{n}{k} = |P| = |P_k| + |P_{k+1}| + \dots + |P_n|$.

Now, any set $A \in P_m$ has the form $A = B \cup \{m\}$ for some $B \subseteq \{1,\dots,m-1\}$ with $|B| = k-1$. Thus, $|P_m|$ is the same as the number of possible B's, which is $\binom{m-1}{k-1}$. Thus, the equation $\binom{n}{k} = |P_k| + \dots + |P_n|$ becomes

$$\binom{n}{k} = \binom{k-1}{k-1} + \binom{k}{k-1} + \dots + \binom{n-1}{k-1} = \sum_{m=k}^{n} \binom{m-1}{k-1} \qquad \square.$$

Example 17:

Consider the sum of everything in the triangle



We divide into N strips as shown. In the p'th strip we have $1+2+\dots+p$. This is $p(p+1)/2 = \binom{p+1}{2}$ by the standard arithmetic progression formula. (There are p terms, & the average is $\frac{p+1}{2}$, giving $p\,\frac{p+1}{2}$ in total). Alternatively, we can take $n=p+1$ & $k=2$ in Example 16 to get

$$\binom{p+1}{2} = \binom{1}{1} + \binom{2}{1} + \dots + \binom{p}{1} = 1+2+\dots+p.$$

Now the overall total is (sum of strip 1) + ... + (sum of strip N)

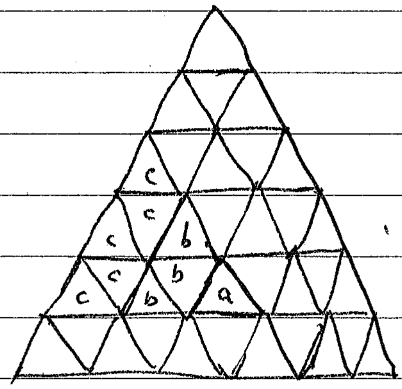
$$= \binom{2}{2} + \binom{3}{2} + \dots + \binom{N+1}{2}$$

By taking $n=N+2$ & $k=3$ in Example 16, we get

$$\binom{N+2}{3} = \binom{2}{2} + \binom{3}{2} + \dots + \binom{N+1}{2} = \text{sum of all numbers in the triangle}$$

RB.

Example 18: Consider this triangle T:

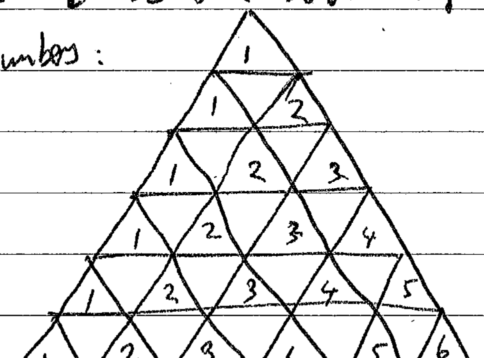


We want to count all the upward pointing subtriangles. For example:

- The triangle marked a is an upward subtriangle of size 1
- The triangles marked a or b together form a triangle a' of size 2, with a as the bottom right corner.
- The triangles marked a, b or c together form a triangle a'' of size 3, again with a as the bottom right corner.

This shows that there are precisely three subtriangles with a as the bottom right corner. This is essentially because a is three steps out from the left hand edge.

We can mark each upward triangle t of size 1 with the number of subtriangles of any size with t as the bottom right corner. We get the following numbers:



These are the same as the numbers in Example 17, so the sum is $\binom{6+2}{3} = \binom{8}{3}$

On the other hand, we can group all the subtriangles by their bottom right corners, & conclude that this sum is just the total number of subtriangles, ie there are $\binom{8}{3}$ subtriangles in total.

This generalises in an obvious way: given a triangle above with N rows, the total number of upward-pointing subtriangles is $\binom{N+2}{3}$.

Example 19:

The Fibonacci sequence $(f_n)_{n\geq 1}$ is defined by $f_1 = 1$, $f_2 = 2$ & $f_{n+2} = f_{n+1} + f_n$ for $n \geq 1$.

The claim is that f_n is the same as the number $g_n = \binom{n}{0} + \binom{n-1}{1} + \binom{n-2}{2} + \cdots = \sum_{k\geq 0} \binom{n-k}{k}$.

(Recall here that $\binom{m}{k}$ is defined to be zero if $m<0$ or $k>m$, so this sum stops after finitely many terms.)

First, we have
$$g_1 = \binom{1}{0} + \binom{0}{1} + \binom{-1}{2} + \cdots = 1+0+0+\cdots = 1 = f_1$$
$$g_2 = \binom{2}{0} + \binom{1}{1} + \binom{0}{2} + \cdots = 1+1+0+\cdots = 2 = f_2.$$

Next, recall Pascal's relation $\binom{m}{k} = \binom{m-1}{k} + \binom{m-1}{k-1}$. Applying this with $m = n-k$ in the definition of g_n gives
$$g_n = \sum_{k\geq 0} \binom{n-k-1}{k} + \sum_{k\geq 0} \binom{n-k-1}{k-1}$$

The first sum directly matches the definition of g_{n-1}. In the second sum, we note that the $k=0$ term is $\binom{n-1}{-1} = 0$. We can therefore start with $k=1$ instead. We can then rewrite the sum in terms of the variable $l = k-1$, so $l \geq 0$ & $k-1 = l$ & $n-k-1 = n-(l+1)-1 = n-2-l$. The second sum then becomes $\sum_{l\geq 0} \binom{n-2-l}{l}$, which is g_{n-2}.

We therefore conclude that $g_n = g_{n-1} + g_{n-2}$, for all n, or equivalently $g_{n+2} = g_{n+1} + g_n$.

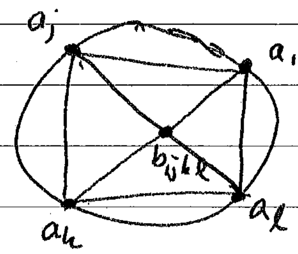
Thus, g_n obeys the same recurrence relation as f_n & we also have $f_1 = g_1$ & $f_2 = g_2$ so we see by induction on n that $f_n = g_n$ for all $n \geq 1$. $\square$.

Example 20:

Suppose we have points $a_1, \dots, a_n$ in order on the circumference of the unit circle, and we draw a line l_{ij} from a_i to a_j for each $i \neq j$. Suppose that these are in general position, so there is no point where more than two of the lines meet.

As $l_{ij} = l_{ji}$, we see that the number of lines is the number of possible subsets $\{i,j\}$ of size 2, ie $\binom{n}{2}$.

Now suppose we have a subset of size 4 in $\{1, \dots, n\}$, say $\{i,j,k,l\}$ with $i<j<k<l$. As we have numbered the points in order around the circle, we find that l_{ik} meets l_{jl} ~~meets~~ at an interior point which we call b_{ijkl}, and that there are no other interior intersection points for the lines l_{ij}, l_{ik}, l_{il}, l_{jk}, l_{jl} and l_{kl}. This construction gives a one to one correspondence between subsets of size 4 in $\{1, \dots, n\}$, and interior intersection points. So the number of such points is $\binom{n}{4}$.



Now suppose we add the lines l_{ij} one by one, keeping track of the numbers L, C & R of lines, internal crossings, and regions. Suppose we add a line l, and that this crosses a certain number p of existing lines, at points $x_1, \dots, x_p$ in order. This means that L increases by 1 & C increases by p. Let x_0 & x_{p+1} be the endpoints of L. Then the segment $[x_i, x_{i+1}]$ splits a region in two, creating one new region. This happens for $i=0, i=1, \dots, i=p$. So we create $p+1$ new regions. This means that $R-C-L$ is unchanged. At the beginning, we have $(L,C,R) = (0,0,1)$. At the end, we have $(L,C,R) = (\binom{n}{2}, \binom{n}{4}, R_{final})$. As $R-C-L$ is unchanged, we have $R_{final} - \binom{n}{4} - \binom{n}{2} = 1-0-0$ ie $R_{final} = 1 + \binom{n}{2} + \binom{n}{4}$ $\square$.