(1)

Example 21:

Consider the equation $2x+6y=11$, which is equivalent to $y=(11-2x)/6$. For any $x \in \mathbb{R}$ we can define $y=(11-2x)/6 \in \mathbb{R}$ to get a solution in real numbers. So there are infinitely many real solutions.

However, if x & y are integers then $2x+6y$ is an even integer & so is not equal to 11. Thus, there are no integer solutions.

Example 22:

(1) If n is odd then an $n \times n$ board has n^2 squares, & n^2 is also odd. Any collection of k disjoint dominos covers $2k$ squares, so cannot cover the whole board, because $2k$ is even & n^2 is odd.

On the other hand, suppose that n is even. Say $n=2m$. Then we can cover an $n \times n$ board with $2m$ rows of m horizontal dominos. (There are also many other more complicated ways to cover an $n \times n$ board with disjoint dominos).

(2) Suppose we start with an $n \times n$ board, & remove two opposite corners. If n was odd then n^2-2 is still odd so we have an odd number of squares & the board still cannot be covered by dominos. Suppose instead that n is even, say $n=2m$. We then find that the two deleted squares have the same colour. Suppose that they are both black. (The case where they are both white is essentially the same.) We then have $2m^2$ white squares & $2m^2-2$ black squares. Any set of k disjoint dominos covers k black squares & k white squares. The number of black & white squares covered is always the same, but our board has different numbers of squares of the two colours, so we cannot cover the whole board.

Example 23:

We can imagine laying down the tiles on a large grid coloured like a chessboard. Each of the first six tiles can be thought of as two dominos stuck together, so it will cover two white squares & two black squares, no matter how it is placed. However, the final T-shaped tile will either cover one white & three black squares, or three white & one black. Thus, the seven tiles together cover either 13 white & 15 black squares, or 15 white & 13 black. However, a 4×7 rectangle always covers 14 squares of each colour. Thus, our tiles cannot be arranged to cover a 4×7 rectangle.

Example 24:

Suppose we have a set of nodes $A_1, \dots, A_n$, with various bridges between them. Suppose that the number of bridges attached to A_i is d_i.

Suppose that there is a tour of the network starting at A_p & ending at A_q (where p might or might not be the same as q)
~~consider a node A_i where $i \neq p, q$~~
Suppose that this tour crosses each bridge precisely once.

Consider a node A_i where $i \neq p$ & $i \neq q$, so A_i is not at the beginning or the end of the tour. Let k_i be the number of times the tour passes through A_i. Each time we visit A_i we arrive on one bridge & leave on another, & all these bridges are different, & we must use every bridge attached to A_i. It follows that $d_i = 2k_i$, & in particular that d_i is even. This holds for all $i \neq p, q$, so it holds for all i with at most two exceptions.

The Königsberg network has nodes A_1, A_2, A_3, A_4 with $d_1 = 3$, $d_2 = 5$, $d_3 = 3$, $d_4 = 3$ so all d_i's are odd. This ~~is~~ would ~~clearly~~ give a contradiction if there was a tour of the type discussed above. So no such tour can exist.

Example 25:

Suppose that some handshakes take place between people $p_1, \ldots, p_n$.
Let d_i be the number of hands shaken by p_i.
We assume that no-one shakes their own hand, or shakes anyone else's hand more than once, so $d_i \in \{0, 1, \ldots, n-1\}$.

The claim is that ~~[illegible]~~ there are two people who shake the same number of hands, or in other words that there exist i & j with $i \neq j$ but $d_i = d_j$. We will assume instead that all the d_i are different, and show that this leads to a contradiction.

If the numbers $d_1, \ldots, d_n$ are all different, then they must completely fill up the set $\{0, 1, \ldots, n-1\}$. Thus, we must have ~~[illegible]~~ $0 = d_u$ for some u, & $n-1 = d_v$ for some v.
As $d_v = n-1$ we see that person p_v must shake everyone else's hand. In particular, p_v must shake p_u's hand, so $d_u > 0$, but $d_u = 0$ by assumption, which is impossible.
Thus the numbers $d_1, \ldots, d_n$ cannot all be different after all.

Example 26:
A typical person has about 10^5 hairs. We can certainly assume that everyone has $< 10^6$ hairs. So we can imagine dividing people into groups $H_0, H_1, \ldots, H_{10^6-1}$ where H_k is the set of people with k hairs. The ~~[illegible]~~ total number of people is then $|H_0| + |H_1| + \cdots + |H_{10^6-1}|$. If ~~[illegible]~~ each set H_k has at most one member, then $|H_k| \leq 1$ for all k so the total number of people is at most $1+1+\cdots+1$ with 10^6 terms, ie 10^6. As there are in fact more than 10^6 people in the world, there must be some set H_k with $|H_k| > 1$. Any two members of H_k will be two different people with the same number of hairs (because they both have k hairs.)

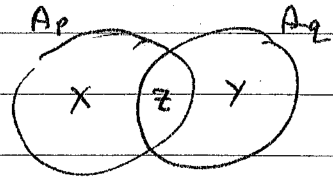
Example 27:

Suppose that $U \subseteq \{1,2,\dots,100\}$ with $|U| = 10$. The claim is that there are two ~~different~~ disjoint subsets of U with the same sum.

As $|U| = 10$, the number of subsets of U is $2^{10} = 1024$. Let the subsets be $A_1, A_2, \dots, A_{1024}$. Let s_k be the sum of the elements in A_k. Note that s_k is the sum of at most 10 terms, and all those terms are different, & each term is at most 100. The largest possible sum like this is $91+92+\dots+100$. By the standard arithmetic progression formula, this is $10 \times \frac{91+100}{2} = 955$. We thus have number $s_1, \dots, s_{1024}$ in the set $S = \{0,1,\dots,955\}$, which has $|S| = 956 < 1024$. Thus, the numbes s_i cannot all be different.

We can therefore find indices p, q with $p \neq q$ such that $s_p = s_q$, ie A_p & A_q have the same sum.

Now put $Z = A_p \cap A_q$ & $X = A_p \setminus A_q$ & $Y = A_q \setminus A_p$ so we have a Venn diagram like



put x = sum of all elements of X,

y = " " " of Y so $s_p = x + z$

z = " " " of Z $s_q = y + z$.

By assumption we have $s_p = s_q$ ie $x + z = y + z$ ie $x = y$. This means that X & Y are disjoint subsets of U with the same sum.

Example 28:
Consider a sequence $x_1, \dots, x_n$ with $x_i \in \mathbb{Z}$.
We claim that there is some consecutive subsequence $x_p, x_{p+1}, \dots, x_q$ (where $p \leq q$) such that $x_p + \dots + x_q = 0 \mod n$.

To prove this, put $y_0 = 0$, $y_1 = x_1 \mod n$, $y_2 = x_1 + x_2 \mod n$, $\dots$ $y_n = x_1 + \dots + x_n \mod n$.

This gives $(n+1)$ numbers $y_0, \dots, y_n$ lying in the set $\{0, \dots, n-1\}$ so they cannot all be different. We can thus find $m < q$ with $y_m = y_q$. Now

$$y_m = x_1 + \dots + x_m \mod n$$
$$y_q = x_1 + \dots + x_m + x_{m+1} + \dots + x_q \mod n$$

Subtracting these, we get $x_{m+1} + \dots + x_q = 0 \mod n$.
We can thus take $p = m+1 \leq q$ & we get $x_p + \dots + x_q = 0 \mod n$ as claimed $\square$.

Example 29
Put x_i = amount deposited on day $i \in \{1, 2\}$
y_i = total deposited up to & including day $i = x_1 + \dots + x_i$ ($\& \ y_0 = 0$)
We are told that $x_i = 1$ fifty times & $x_i = 2$ fifty times so the total deposited is $50 + 50 \times 2 = 150$ so $y_{100} = 150$.
So $0 \leq y_i \leq 150$ for all i. Also $y_{i+1} = y_i + x_{i+1} > y_i$ so the numbers y_i are all distinct. Put $Y = \{y_0, \dots, y_{100}\}$ so $|Y| = 101$. & $Y \subseteq \{0, \dots, 150\}$
Now fix k with $1 \leq k < 50$ & put $z_i = y_i + k$ & $Z = \{z_0, \dots, z_{100}\}$ so $|Z| = 101$ & $Z \subseteq \{k, \dots, 150+k\} \subseteq \{0, \dots, 199\}$. Now both Y & Z are contained in the set $U = \{0, \dots, 199\}$ with $|U| = 200$ & $|Y| + |Z| = 202 > |U|$. It follows that Y & Z cannot be disjoint. Thus, we have $z_i = y_j$ for some i, j, which means that $y_i + k = y_j$. As the sequence y is increasing, we must have $i < j$.
Now $x_{i+1} + \dots + x_j = y_j - y_i = k$, so we deposit a total of k over days $i+1$ to j $\square$.

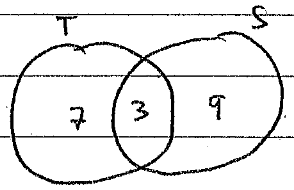
Example 30:

3 people play both T & S

10 people play T. Of these, 3 play S as well. So 7 play T only

12 " " S. Of these, 3 play T as well. So 9 play S only.

The total who play T or S is thus $3 + 7 + 9 = 19$



Example 31:

2 people play T, S & B (1)

3 play S & B ; of these 2 play T as well, so 1 plays S&B not T (2)

4 play T & B ; " " 2 play S " " , " 2 play T&B not S (3)

5 play T & S ; " " 2 play B " " , " 3 play T&S not B (4)

Total playing T + something else $= \overset{(1)}{2} + \overset{(3)}{2} + \overset{(4)}{3} = 7$ (5)

Total playing T $= 10$ $\therefore$ total playing T only $= 10 - \overset{(5)}{7} = 3$. (6)

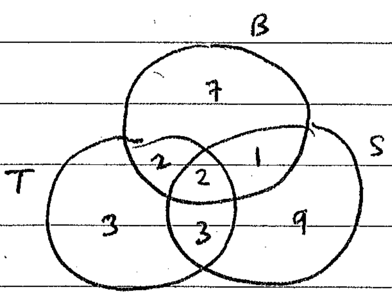
Total playing S + something else $= \overset{(1)}{2} + \overset{(2)}{1} + \overset{(4)}{3} = 6$ (7)

Total playing S $= 15$ $\therefore$ total playing S only $= 15 - \overset{(7)}{6} = 9$ (8)

Total playing B + something else $= \overset{(1)}{2} + \overset{(2)}{1} + \overset{(3)}{2} = 5$ (9)

Total playing B $= 12$ $\therefore$ total playing B only $= 12 - \overset{(9)}{5} = 7$ (10)

Total playing T, S or B $= \overset{(1)}{2} + \overset{(2)}{1} + \overset{(3)}{2} + \overset{(4)}{3} + \overset{(6)}{3} + \overset{(8)}{9} + \overset{(10)}{7} = 27$



Example 30 using the Inclusion-Exclusion Principle:

$|T \cup S| = |T| + |S| - |T \cap S| = 10 + 12 - 3 = 19$

Example 31 using the Inclusion-Exclusion Principle:

$|T \cup S \cup B| = |T| + |S| + |B| - |T \cap S| - |T \cap B| - |S \cap B| + |T \cap S \cap B|$

$= 10 + 15 + 12 - 5 - 4 - 3 + 2$

$= 27$

Statement of the Inclusion-Exclusion principle:

Suppose we have a set B, and subsets $B_1, \ldots, B_n \subseteq B$

We want to calculate $|B^*|$ where $B^* = B_1 \cup \cdots \cup B_n$

or maybe $|B \setminus B^*| = |B| - |B^*|$.

For any set $I \subseteq \{1, \ldots, n\}$ we put $B_I = \bigcap_{i \in I} B_i$

Eg $B_{\{i\}} = B_i$, $B_{\{i,j\}} = B_i \cap B_j$, $B_{\{i,j,k\}} = B_i \cap B_j \cap B_k$ etc.

We interpret this as $B_\emptyset = B$ for the case $I = \emptyset$.

Claim: $|B^*| = \sum_{I \neq \emptyset} (-1)^{|I|+1} |B_I| = |B_1| + \cdots + |B_n| - |B_1 \cap B_2| - \cdots - |B_{n-1} \cap B_n|$
$+ |B_1 \cap B_2 \cap B_3| \cdots \pm |B_1 \cap \cdots \cap B_n|$

$$|B \setminus B^*| = \sum_I (-1)^{|I|} |B_I| = |B| - |B_1| - \cdots - |B_n| + |B_1 \cap B_2| \cdots$$

Proof: if $n=0$ then $B^* = \emptyset$, the sum over I has no terms, $0=0$ ✓.

if $n=1$: $B^* = B_1$, the only term in the sum is for $I = \{1\}$, the claim is that $|B_1| = |B_1|$ which is true.

If $n=2$: the claim is that $|B_1 \cup B_2| = |B_1| + |B_2| - |B_1 \cap B_2|$

Put $B_i' = B_i \setminus (B_1 \cap B_2)$ so B_i is the disjoint union of $B_1 \cap B_2$ & B_i', whereas $B_1 \cup B_2$ is the disjoint union of $B_1 \cap B_2$ and B_1' and B_2'.

we therefore have $|B_1| = |B_1'| + |B_1 \cap B_2|$

$$|B_2| = |B_2'| + |B_1 \cap B_2|$$

$$\begin{aligned} |B_1 \cup B_2| &= |B_1'| + |B_2'| + |B_1 \cap B_2| \\ &= (|B_1'| + |B_1 \cap B_2|) + (|B_2'| + |B_1 \cap B_2|) - |B_1 \cap B_2| \\ &= |B_1| + |B_2| - |B_1 \cap B_2| \end{aligned}$$

as claimed.

Now suppose that $n > 2$. ~~By induction we can assume~~ ~~then~~ Put $X = B_1 \cup \dots \cup B_{n-1}$ & $Y = X \cap B_n$ so $B^* = X \cup B_n$. By the $n=2$ case we have $|B^*| = |X| + |B_n| - |Y|$.

We can assume by induction that the IEP works for any list of $n-1$ sets, such as $B_1, \dots, B_{n-1}$. This tells us that $|X|$ is the sum of terms $(-1)^{|I|-1}|B_I|$ for $I \subseteq \{1,\dots,n-1\}$ with $I \neq \emptyset$.

Next, we can think of Y as $C_1 \cup \dots \cup C_{n-1}$, where $C_i = B_i \cap B_n$. This shows that $|Y|$ is the sum of terms $(-1)^{|J|-1}|C_J|$, where J runs over subsets $J \subseteq \{1,\dots,n-1\}$ with $J \neq \emptyset$. Here if $J = \{j_1,\dots,j_r\}$ we have

$$\begin{aligned} C_J = C_{j_1} \cap \dots \cap C_{j_r} &= B_{j_1} \cap B_n \cap \dots \cap B_{j_r} \cap B_n \\ &= B_{j_1} \cap \dots \cap B_{j_r} \cap B_n = B_{J \cup \{n\}} \end{aligned}$$

We can rewrite this in terms of the sets $I = J \cup \{n\}$, so $(-1)^{|J|-1} = (-1)^{|I|}$ & $|C_J| = |B_I|$ & the condition $J \neq \emptyset$ becomes $I \neq \{n\}$. We find that $-|Y| = \sum_{n \in I,\, I \neq \{n\}} (-1)^{|I|-1}|B_I|$

For the exceptional case $I = \{n\}$, we also have $(-1)^{|I|-1}|B_I| = +|B_n|$

Putting this together: the terms with $n \notin I$ match up with $|X|$, the term with $I = \{n\}$ matches with $|B_n|$, & the terms with $n \in I$ & $I \neq \{n\}$ match with $-|Y|$, so the equation $|B^*| = |X| + |B_n| - |Y|$ becomes $|B^*| = \sum_{\substack{I \subseteq \{1,\dots,n\} \\ I \neq \emptyset}} (-1)^{|I|-1}|B_I|$. This proves the induction step. $\square$

Alternative proof of the IEP.:

Lemma: consider a finite set I. Then $\sum_{J \subseteq I} (-1)^{|J|} = \begin{cases} 1 & \text{if } I = \emptyset \\ 0 & \text{if } I \neq \emptyset \end{cases}$

Proof: If $I = \emptyset$ then the only possible J is $J = \emptyset$ so the sum is $(-1)^{|\emptyset|} = (-1)^0 = 1$.

Suppose instead that $I \neq \emptyset$, so we can choose $u \in I$. Put $I' = I \setminus \{u\}$. ~~Any subset $J \subseteq I$ is either~~ For any $J' \subseteq I'$ we have two subsets of I, namely J' & $J' \cup \{u\}$, and every subset of I arises like this. So
$$\sum_{J \subseteq I} (-1)^{|J|} = \sum_{J' \subseteq I'} \left((-1)^{|J'|} + (-1)^{|J' \cup \{u\}|} \right)$$

but $|J' \cup \{u\}| = |J'| + 1$ so the two terms cancel so $\sum_{J \subseteq I} (-1)^{|J|} = 0$ □

We now have $\sum_{I \subseteq \{1,\dots,n\}} (-1)^{|I|} |B_I| = \sum_{I \subseteq \{1,\dots,n\}} \sum_{b \in B_I} (-1)^{|I|}$

For any $b \in B$ put $T_b = \{i : b \in B_i\}$ so
$b \in B_I \iff$ (for all $i \in I$, $b \in B_i$)
$\iff$ (for all $i \in I$, $i \in T_b$) $\iff I \subseteq T_b$.
Note that $T_b = \emptyset \iff b$ is not in any $B_i \iff b \in B \setminus B^*$
The sum can now be written as the sum of terms $(-1)^{|I|}$ for pairs (I, b) with $b \in B_I$, or equivalently pairs (I, b) with $I \subseteq T_b$, so we get $\sum_b \sum_{J \subseteq T_b} (-1)^{|J|}$.

The lemma tells us that the inner sum is zero if $T_b \neq \emptyset$, & one if $T_b = \emptyset$. In other words, the inner sum is one if $b \in B \setminus B^*$, & zero otherwise. Thus, the outer sum is $|B \setminus B^*|$.
In other words, $|B \setminus B^*| = \sum_I (-1)^{|I|} |B_I|$ as claimed. □

Example 33:

Put $B = \{\text{permutations of } 1,\dots,n\}$

$B_i = \{\text{permutations that send } i \text{ to } i\}$

so $B' = B_1 \cup \dots \cup B_n = \{\text{permutations that have a fixed point}\}$

$B \setminus B' = \{\text{permutations with no fixed points}\} = \{\text{derangements}\}$.

The IEP gives $\#\text{ derangements} = \sum_{I \subseteq \{1,\dots,n\}} (-1)^{|I|} |B_I|$

Here B_I is the set of permutations that act as the identity on I, but permute the set $\{1,\dots,n\} \setminus I$ (of size $n-|I|$) arbitrarily. It follows that $|B_I| = (n-|I|)!$

Also, if we fix k then there are $\binom{n}{k}$ possible choices of I with $|I|=k$, & for each such I we have $|B_I| = (n-k)!$

This gives $\#\text{ derangements} = \sum_{k=0}^{n} (-1)^k \binom{n}{k} (n-k)!$

Now $\binom{n}{k} = \frac{n!}{k!\,(n-k)!}$ so this simplifies to $n! \sum_{k=0}^{n} (-1)^k/k!$

The total number of permutations of $\{1,\dots,n\}$ is $n!$, so the proportion of derangements is $p_n = \sum_{k=0}^{n} (-1)^k/k!$.

Recall also that $e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!}$ so $e^{-1} = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!}$

so $p_n \to e^{-1}$ as $n \to \infty$.

Example 34:

Note that $42 = 2\times3\times7$, so a number b is coprime with 42 iff it is not divisible by 2, 3 or 7.

0^{237}	6^{23}	12^{23}	18^{23}	24^{23}	30^{23}	36^{23}
1	7^{7}	13	19	25	31	37
2^{2}	8^{2}	14^{27}	20^{2}	26^{2}	32^{2}	38^{2}
3^{3}	9^{3}	15^{3}	21^{37}	27^{3}	33^{3}	39^{3}
4^{2}	10^{2}	16^{2}	22^{2}	28^{27}	34^{2}	40^{2}
5	11	17	23	29	35^{7}	41

Here we have written the numbers $0,\dots,41$. We have written a 2, 3 or 7 next to every number that is divisible by 2, 3 or 7. The ~~next~~ ringed numbers have no 2, 3 or 7 so they are coprime with 42. There are 12 of them.

~~Example~~

Theorem 35:

More generally, suppose that $m>1$, and that $p_1,\dots,p_n$ are the distinct prime divisors of m. Put $B=\{0,\dots,m-1\}$ & $B_i = \{b\in B : p_i \text{ divides } b\}$. ~~...~~ Put $U = \{b \in B : b \,\&\, m \text{ are coprime}\}$ so $U = B\setminus(B_1\cup\dots\cup B_n)$ so IEP gives

$$|U| = \sum_{I\subseteq\{1,\dots,n\}} (-1)^{|I|}|B_I|.$$

Now B_I is the set of numbers $b\in B$ that are divisible by $\prod_{i\in I} p_i$, so $|B_I| = m/\prod_{i\in I}p_i$, so $(-1)^{|I|}|B_I| = m\prod_{i\in I}\frac{-1}{p_i}$

$$|U| = m\sum_{I\subseteq\{1,\dots,n\}}\prod_{i\in I}\left(\frac{-1}{p_i}\right) = m\left(1-\frac{1}{p_1}\right)\dots\left(1-\frac{1}{p_n}\right) \quad \square$$