①

Consider a job allocation problem:

- Finite set J of jobs
- Finite set P of people
- For each $j \in J$, a subset $C[j] \subseteq P$ of candidates, ie people qualified to do job j
- Corresponds to a board with rows labelled by J, columns labelled by P, spot (j,p) is unblocked iff $p \in C[j]$.
- Given a subset $U \subseteq J$ of jobs, a *partial matching* on U is an injective map $m : U \longrightarrow P$ with $m(j) \in C[j]$ for all j.

  NB: • $m(j) \in C[j]$ means we assign j to a person $m(j)$ who is qualified to do j
  • injective means we do not assign two different jobs to the same person.
- A full matching is a map m as above with $U = J$
- Rook polynomials count partial matchings.

  The Rook IEP theorem can help count full matchings.
- Here study a simpler problem: is there at least one full matching?

  If so, say that the problem is solvable.

- Defn: $C[U] = \bigcup_{j \in U} C[j]$ eg $C[\{j,k,l\}] = C[j] \cup C[k] \cup C[l]$

  $C[U] = \{$people qualified to do at least one of the jobs in $U\}$.

Defn: U is implausible if $|C[U]| < |U|$

barely plausible if $|C[U]| = |U|$

very plausible if $|C[U]| > |U|$

plausible if $|C[U]| \geq |U|$ (ie barely plausible or very plausible)

Lemma: if there is a full matching $m : J \to P$, then every subset $U \subseteq J$ is plausible.

Proof: Suppose $U = \{j_1, \ldots, j_r\}$. Then $m(j_i) \in C[j_i] \subseteq C[U]$ for all i, & $m(j_1), \ldots, m(j_r)$ are all different as m is injective $\therefore |C[U]| \geq r = |U|$ $\square$

Corollary: if there are any implausible sets, then the problem is not solvable.

②

Eg

	Paula	Quinn	Ruth	Steve	Tessa
Artist	✓	✓			
Baker		✓	✓		✓
Courier	✓	✓			
Dentist	✓		✓	✓	✓
Electrician	✓	✓			

$C[\{A\}] = \{P, Q\}$ so $\{A\}$ is very plausible
$C[\{B,D\}] = \{P,Q,R,S,T\}$ so $\{B,D\}$ is very plausible
$C[\{A,C\}] = \{P,Q\}$ so $\{A,C\}$ is barely plausible
$C[\{A,C,E\}] = \{P,Q\}$ so $\{A,C,E\}$ is implausible $\therefore$ problem not solvable.

Hall's Marriage ~~Theorem~~ Theorem: the converse also holds, ie
if every set of jobs is plausible, then there exists a full matching.

First proof:
- if there are no jobs, there is nothing to do & the problem is vacuously solved.

~~If there is only one job j, then there exists~~
- Suppose there is only one job j. By assumption, the set $\{j\}$ is plausible, ie $|C[j]| \geq |\{j\}| = 1$, so we can choose $p \in C[j]$. We can then define a full matching by $m(j) = p$.

- Suppose that $|J| = n > 1$. By induction, we can assume that any matching problem with less than n jobs & all subsets plausible can be solved.

We need to consider two cases.
(α) There is a subset J_0 with $\emptyset \neq J_0 \neq J$ that is barely plausible.
(β) Every subset with $\emptyset \neq U \neq J$ is very plausible.

(3)

First consider case (β). Choose any job $j_0 \in J$ & put $J_1 = J \setminus \{j_0\}$.
As $\{j_0\}$ is plausible we see that $C[j_0] \neq \emptyset$ so we can choose $p_0 \in C[j_0]$.
Now for any $j \in J_1$ put $C_1[j] = C[j] \setminus \{p_0\}$.
This defines a matching problem for J_1, where we try to assign to each $j \in J_1$ a person who is qualified & is not equal to p_0.
For any $U \subseteq J_1$ we have $C_1[U] = C[U] \setminus \{p_0\}$
If $p_0 \in C[U]$ then $|C_1[U]| = |C[U]| - 1$;
if $p_0 \notin C[U]$ then $|C_1[U]| = |C[U]|$
either way we have $|C_1[U]| \geq |C[U]| - 1$.
As all sets are very plausible, we have $|C[U]| > |U|$ & $\therefore$ $|C_1[U]| \geq |U|$
(Edge cases: the inequality $|C[U]| > |U|$ is only for $\emptyset \neq U \neq J$.
As $U \subseteq J_1$ here we have $U \neq J$. If $U = \emptyset$ then $C_1[U] = \emptyset$ & $|C_1[U]| = |U| = 0$.)

We now see that all subsets are plausible for C_1. By the induction hypothesis, there is a full matching $m_1 : J_1 \to P$. We can extend this over J by putting $m(j_0) = p_0$. This is still injective because $m_1(j) \in C[j] \setminus \{p_0\}$ for all $j \in J_1$, ie $m_1(j) \neq p_0$. This gives a full matching for the original problem.

Now consider case α. Put $J_0 = J \setminus J_1$ so $J = J_0 \cup J_1$.
As $\emptyset \neq J_1 \neq J$ we have $|J_0|, |J_1| < |J|$ so every plausible matching problem on J_0 or J_1 is solvable. In particular, we can restrict our original problem to J_1 & choose $m_1 : J_1 \to P$ which solves it.
Note that $m_1(J_1) \subseteq C[J_1]$ & m_1 is injective so $|m_1(J_1)| = |J_1|$. Also J_1 is barely plausible by assumption so $|C[J_1]| = |J_1|$ so we must have $m_1(J_1) = C[J_1]$.

We could just do the same thing & choose $m_0 : J_0 \to P$ solving the restriction of the original problem to J_0. However, this does not work properly: the images $m_0(J_0)$ & $m_1(J_1)$ might overlap, & then the combined map $J \to P$ would not be injective.
So we need to do something a bit more subtle.

For each $j \in J_0$ we put $C_0[j] = C[j] \setminus C[J_1] = C[j] \setminus m_1(J_1)$

Then for $U \subseteq J_0$ we have $C_0[U] = C[U] \setminus C[J_1]$.
Note that this is the same as $C[U \cup J_1] \setminus C[J_1]$, because it does not matter if we put in the sets $C[J_1]$ & then take them out again.
As the original problem has all sets plausible, we know that
$|C[U \cup J_1]| \geq |U \cup J_1| = |U| + |J_1| = |U| + |C[J_1]|$
& $\therefore$ $|C_0[U]| \geq |C[U \cup J_1]| - |C[J_1]| \geq |U| + |C[J_1]| - |C[J_1]| = |U|$.

Thus, all sets are plausible for the new problem C_0 on J_0.
By the induction hypothesis, we can choose a full matching $m_0 : J_0 \rightarrowtail P$ with $m_0(j) \in C_0[j] = C[j] \setminus m_1(J_1)$ for all $j \in J_0$.
As $m_0(j) \notin m_1(J_1)$ we see that the combined map $m : J = J_0 \cup J_1 \longrightarrow P$ is injective & is therefore a full matching for the original problem. $\square$.

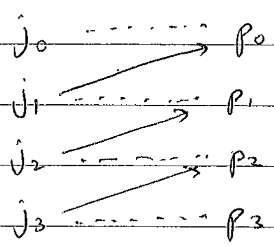
Second proof:

Suppose we have a ^plausible^ matching problem as before, a partial matching $M_\varphi : J_\varphi \to P$ (for some $J_\varphi \subset J$) and another job $j_0 \notin J_\varphi$.
We want to assign job j_0 as well, but this may not be possible without changing the assignments that we have already made.

~~[illegible] a [illegible] zigzag means a pair of sequences~~

Example:

j_0 has not been assigned
p_0 is qualified for j_0, but has already been assigned to j_1
p_1 " " " j_1, " " " to j_2
p_2 " " " j_2, " " " to j_3
p_3 " " " j_3, & has not yet been given a job



new assignment
$j_0 \longrightarrow p_0$
$j_1 \longrightarrow p_1$
$j_2 \longrightarrow p_2$
$j_3 \longrightarrow p_3$

(dotted line : person is qualified
solid line : person is assigned (& is also qualified))

A sequence $(j_0, \dots, j_n ; p_0, \dots, p_n)$ like this (where p_n has not been assigned a job) is called an <u>open zigzag</u>. The process of changing the assignment as above is called <u>flipping</u>. If we can find an open zigzag then we can flip it, which ~~extends our matching~~ modifies & extends m, so it covers j_0 as well as J_φ.

Problem: is there always an open zigzag?

Defn: a closed zigzag is a sequence $(j_0,\dots,j_n,p_0,\dots,p_{n-1})$ such that
(0) j_0 is the extra job that we want to assign
(1) $j_1,\dots,j_n$ are all different and lie in the set J_1 that we have already assigned
(2) $m_1(j_1)=p_0$, $m_1(j_2)=p_1,\dots,m_1(j_n)=p_{n-1}$ (& ∴ $p_0,\dots,p_{n-1}$ all different)
(3) $p_0\in C[j_0],\dots,p_n\in C[j_n]$

If we have a closed zigzag $(j_0,\dots,j_n,p_0,\dots,p_{n-1})$ then we may or may not be able to find $p_n\in C[j_n]$ different from $p_0,\dots,p_{n-1}$.
If we can find p_n, then it might or might not lie in $m_1(J_1)$ (ie p_n might or might not have been assigned a job already).
If $p_n\notin m_1(J_1)$ then we have an open zigzag that we can flip.
If $p_n\in m_1(J_1)$ then we have $p_n=m_1(j_{n+1})$ say & necessarily $j_{n+1}\notin\{j_0,\dots,j_n\}$ so we have a new closed zigzag $(j_0,\dots,j_{n+1},p_0,\dots,p_n)$.

Let U be the set of jobs that occur in some closed zigzag.
The sequence (j_0) by itself counts as a closed zigzag with $n=0$, so $j_0\in U$. ~~Also~~ By condition (1) in the definition, all other elements of U lie in J_1. So $U=\{j_0\}\cup U_1$ for some $U_1\subseteq J_1$.

By assumption, our matching problem is plausible, so
$|C[U]|\geqslant|U|=1+|U_1|$ but $m_1(U_1)\subseteq C[U]$ & m_1 is injective so $|m_1(U_1)|=|U_1|$ so $|C[U]\setminus m_1(U_1)|\geqslant 1$ so we can choose $p^*\in C[U]\setminus m_1(U_1)$. As $p^*\in C[U]$ we can find $j^*\in U$ with $p^*\in C[j^*]$. As $j^*\in U$ we can find a closed zigzag $(j_0,\dots,j_n=j^*,p_0,\dots,p_{n-1})$. As $p^*\in C[j^*]=C[j_n]$ we can extend this by defining $p_n=p^*$. We claim that this gives an open zigzag. [NB p_n is different from $p_0,\dots,p_{n-1}$, as they all lie in $m_1(J_1)$]

We will prove this by contradiction. Suppose that the zigzag is not open, which means that p_n has already been assigned a job, say $p_n=m_1(j_{n+1})$. Then $(j_0,\dots,j_{n+1},p_0,\dots,p_n)$ is a closed zigzag, so j_{n+1} occurs in a closed zigzag, ie $j_{n+1}\in U_1$, so $p_n=m_1(j_{n+1})\in m_1(U_1)$. But $p_n\notin m_1(U_1)$ by definition. This contradiction means we have an open zigzag after all.

To recap: if we have a plausible job allocation problem, a partial solution $m_1 : J_1 \longrightarrow P$, and an extra job $j_0 \in J \setminus J_1$, then we can always find an open zigzag & flip it to get a partial solution $m : J_1 \cup \{j_0\} \longrightarrow P$.

By doing this repeatedly, we can find a full matching for the original problem.

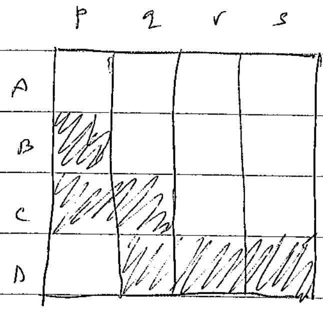
Example: $J = \{A, B, C, D\}$ $\quad P = \{p, q, r, s\}$

$C[A] = \{p, q, r, s\}$

$C[B] = \{q, r, s\}$

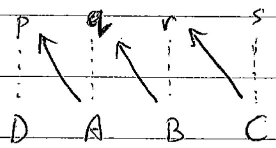
$C[C] = \{r, s\}$

$C[D] = \{p\}$.



For A, B, C we make the obvious choices: $A \mapsto p$, $B \mapsto q$, $C \mapsto r$.
But now we cannot assign D without modifying the ~~stade~~ earlier assignments.

We can find an open zigzag:



& flip to get a complete matching: $(D \mapsto p,\ A \mapsto q,\ B \mapsto r,\ C \mapsto s)$.

Now consider the case where $|P| > |J|$, so that some people will not be assigned a job. Suppose we have a subset $T \subseteq P$ of people who are poor & so really need a job.

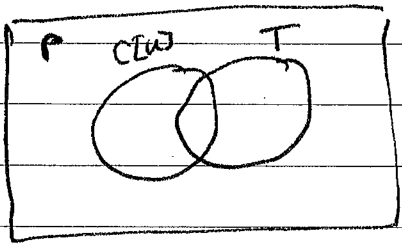
54 Theorem: suppose that every subset $U \subseteq J$ is plausible, ie $|C[U]| \geq |U|$. Suppose also that every U has $|C[U] \cap T| \geq |U| + |T| - |J|$. Then there is a full matching in which every poor person has a job.

Proof: choose a set K of ~~jobs~~ extra jobs that can only be done by rich people, with $|K| = |P| - |J|$.
Put $J^* = J \cup K$, so $|J^*| = |P|$.
As each job $k \in K$ can only be done by rich people, we define $C^*[k] = P \setminus T$ for all $k \in K$. Also $C^*[j] = C[j]$ for $j \in J$.
This gives a new job allocation problem (J^*, P, C^*).

We claim that all subsets $U^* \subseteq J^*$ are plausible for this new problem. There are two cases:

(α) Suppose $U^* \subseteq J$. Then $C^*[U^*]$ is just the same as $C[U^*]$, & we assumed that all sets are plausible for the original problem, so $|C^*[U^*]| \geq |U^*|$.

(β) Suppose instead that $U^* \not\subseteq J$, so $U^* = U \cup V$ for some $U \subseteq J$ & some $\emptyset \neq V \subseteq K$, so $|V| \leq |K| = |P| - |J|$.
As $V \neq \emptyset$ & $C^*[k] = P \setminus T$ for all $k \in K$ we see that $C^*[U^*] = C[U] \cup (P \setminus T)$. (In other words, the candidates for U^* are the candidates for U plus the rich people). By considering the Venn diagram



we see that $C[U] \cup (P \setminus T)$ is the disjoint union of $(C[U] \cap T)$ with $P \setminus T$.

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Thus $|C^+[U^+]| = |C[U]\cap T| + |P| - |T|$

$\geq |U| + \cancel{|T|} - |J| + |P| - \cancel{|T|}$ by assumption

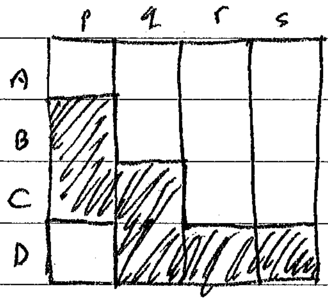
$\geq |U| + |K| \geq |U| + |V| = |U^+|$.

This shows that every subset is plausible for C^+, so Hall's Theorem tells us that there exist a matching $m^+ : J^+ = J \cup K \rightarrowtail P$.

As $|J\cup K| = |P|$, this must be bijective, so everyone has a job. The extra jobs in K can only be assigned to rich people, so every poor person is assigned one of the jobs from the original set J. Thus, if we just ignore the extra jobs in K, we have a full matching $m : J \to P$ in which every poor person has a job. $\square$

Definition: in a job allocation problem as before, for each person p put $Q[p] = \{\text{jobs that } p \text{ can do}\} = \{j \in J \mid p \in C[j]\}$

Eg



$C[A] = \{p,q,r,s\}$ $\quad$ $Q[p] = \{A, D\}$

$C[B] = \{q,r,s\}$ $\quad$ $Q[q] = \{A, B\}$

$C[C] = \{r,s\}$ $\quad$ $Q[r] = \{A,B,C\}$

$C[D] = \{p\}$ $\quad$ $Q[s] = \{A,B,C\}$.

55 Proposition: suppose there is a constant $d > 0$ such that

(a) For all jobs j there are at least d candidates ie $|C[j]| \geq d$

(b) Each person p can do at most d jobs ie $|Q[p]| \leq d$.

Then every set of jobs is plausible, so there is a full matching (by Hall's Theorem).

Proof: ~~Put~~ Consider a set $\mathcal{U}$ of jobs, and put

$$X = \{(j,p) \in \mathcal{U} \times P \mid p \text{ can do } j\} \quad ①$$
$$= \{(j,p) \mid j \in \mathcal{U} \text{ and } p \in C[j]\} \quad ②$$
$$= \{(j,p) \mid p \in P \text{ and } j \in \mathcal{U} \cap Q[p]\} \quad ③$$

From description ② we get $|X| = \sum_{j \in \mathcal{U}} |C[j]| \geq \sum_{j \in \mathcal{U}} d = d|\mathcal{U}|$.

Consider instead description ③. If $p \notin C[\mathcal{U}]$ then p is not qualified to do any of the jobs in $\mathcal{U}$ & $\therefore$ $U \cap Q[p] = \emptyset$. If $p \in C[\mathcal{U}]$ then $U \cap Q[p] \subseteq Q[p]$ & $\therefore$ $|U \cap Q[p]| \leq |Q[p]| \leq d$. Thus $|X| = \sum_{p \in P} |U \cap Q[p]| = \sum_{p \in C[\mathcal{U}]} |U \cap Q[p]| \leq d\,|C[\mathcal{U}]|$.

We now have $d\,|\mathcal{U}| \leq |X| \leq d\,|C[\mathcal{U}]|$ so $|\mathcal{U}| \leq |C[\mathcal{U}]|$ so $\mathcal{U}$ is plausible. $\square$.

Special case: if $|C[j]| = |Q[p]| = d$ for all j & p then there is a full matching.

Corollary 56:

Prop: suppose

- For all j $|C[j]| = d$
- For all p $|Q[p]| \leq d$

& put $T = \{p \mid |Q[p]| = d\} = \{\text{talented people}\}$.

Then there is a full matching in which all talented people get jobs.

Proof: by Cord 55: all sets are plausible $\therefore$ there is a full matching.
By Theorem 54', need to check the additional condition
$|C[U] \cap T| \geq |U| + |T| - |J|$.

Now $|C[U] \cap T| = |T| - |T \setminus C[U]|$
so it is equivalent to check $|T| - |T \setminus C[U]| \geq |U| + |T| - |J|$
or $|T \setminus C[U]| \leq |J| - |U| = |J \setminus U|$.

Put $Y = \left\{ (j,p) \;\middle|\; \begin{array}{l} p \text{ is talented \& qualified for } j \\ \text{but } p \text{ is not qualified for any job in } U \end{array} \right\}$.

Then $|Y| = \sum_{p \in T \setminus C[U]} |Q[p]| = d\,|T \setminus C[U]|$
(as $|Q[p]| = d$ for $p \in T$).

On the other hand, if $(j,p) \in Y$ then $j \notin U$ $\therefore$ $j \in J \setminus U$
so $|Y| \leq \sum_{j \in J \setminus U} |C[j]| = d\,|J \setminus U|$

Now $d\,|T \setminus C[U]| \leq |Y| \leq d\,|J \setminus U|$ $\therefore$ $|T \setminus C[U]| \leq |J \setminus U|$
as required $\square$.

Transversal version:

Suppose we have finite sets $A_1, \dots, A_r$.
A distinct set of representatives is a list $a_1, \dots, a_r$ with all a_i's different & $a_i \in A_i$.

Theorem: there is a distinct set of representatives iff for any set $\{i_1, \dots, i_k\}$ of indices we have $|A_{i_1} \cup \dots \cup A_{i_k}| \geq k$.

Proof: take $J = \{1, \dots, r\}$, $P = A_1 \cup \dots \cup A_r$,
$C[j] = A_j \subseteq P$ so $C[\{i_1, \dots, i_k\}] = A_{i_1} \cup \dots \cup A_{i_k}$.
This gives a matching problem. The condition $|A_{i_1} \cup \dots \cup A_{i_k}| \geq k$ means precisely that each set $\{i_1, \dots, i_k\} \subseteq J$ is plausible.
A distinct set of representatives is the same thing as a full matching.
So this is the same as Hall's theorem. $\square$.

Example 58:
$A_1 = \{1,3\}$ $A_2 = \{2,3\}$ $A_3 = \{1,3,4,5\}$ $A_4 = \{3,4,6\}$ $A_5 = \{1,5\}$ $A_6 = \{1,2\}$
$a_1 = 1$ $a_2 = 3$ $a_3 = 4$ $a_4 = 6$ $a_5 = 1$ $a_6 = 2$
gives a complete set of representatives.

$A_1 = \{1,2,3\}$ $A_2 = \{2,3\}$ $A_3 = \{3,5,7\}$ $A_4 = \{1,2\}$ $A_5 = \{1,2,3\}$ $A_6 = \{4,5,6\}$

We have $A_1 \cup A_2 \cup A_4 \cup A_5 = \{1,2,3\}$
ie for $U = \{1,2,4,5\}$ we have $|C[U]| = 3 < |U|$
$\therefore$ U is not plausible so there is no full matching / distinct set of representatives.

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Team version:

Suppose we again have a set J of jobs, a set P of people, & for each job j we have a subset $C[j] \subseteq P$ of candidates. However, suppose we now need a team of people for each job, a team of size m_j for job j say.

~~Corollary~~ Say $U \subseteq J$ is plausible (for the team problem) iff $|C[U]| \geq \sum_{u \in U} m_u$

(eg $\{j_1, j_2, j_3\}$ is plausible iff $|C[\{j_1, j_2, j_3\}]| \geq m_{j_1} + m_{j_2} + m_{j_3}$)

Prop: allocation is possible iff every subset $U \subseteq J$ is plausible.

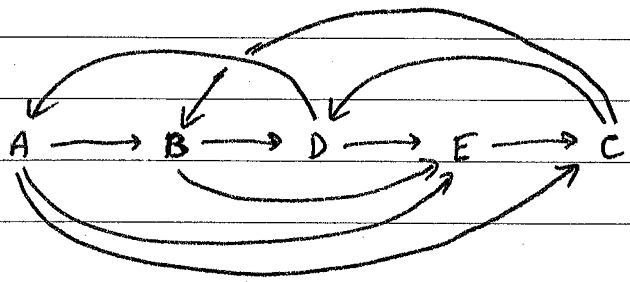
Proof: make a set B of badges, with badges numbered $1, \ldots, m_j$ for job j. If b is a badge for job j then put $D[b] = C[j] = \{$people qualified to wear badge $b\}$. Our problem is just to allocate the badges to distinct, qualified people. ~~So~~ Consider a set V of badges, & let U be the set of jobs appearing in V, so $D[V] = C[U]$. Clearly $|V| \leq |$all badges for $U| = \sum_{u \in U} m_u$ & $\sum_{u \in U} m_u \leq |C[U]|$ by assumption so $|D[V]| \geq |V|$. Thus all sets are plausible for the badge allocation problem, so Hall's Theorem tells us that there is a perfect matching $\square$.

Consider a set P of n players. A tournament on P consists of $\binom{n}{2}$ games in which each player plays once against every other player, & there is one winner for every game

Eg: players A, B, C, D, E ; underline the winner in each game

AB AC AD AE BC BD BE CD CE DE

Or draw arrows from winners to losers:



Scores table :	
A : 3	
B : 2	
C : 2	
D : 2	
E : 1	

score sequence : 32221
(always written in descending order).

NB $A \to B \to D \to E \to C$

Defn : a winning line is a sequence $p_1, \ldots, p_n$ containing each player exactly once, such that
p_1 beats p_2, p_2 beats p_3, ..., p_{n-1} beats p_n.

Lemma: for any tournament of n players, there exists a winning line.

Proof: clear for 0, 1 or 2 players.
For larger n, argue by induction. Choose any one player p^*.
By induction hypothesis, the other players can be ordered as
$p_1, \ldots, p_{n-1}$ where p_i beats p_{i+1}.
If p^* beats p_1, put p^* at the beginning.
If p^* ~~beats~~ no-one, put them at the end.
Otherwise, let ~~p_{2n}~~ p_i be the last player ~~...~~ in the sequence that beats p^*, so p^* beats p_{i+1}.

Then $p_1, \ldots, p_i, p^*, p_{i+1}, \ldots, p_{n-1}$ is a winning line.

Lemma: for $0 \le k \le n$ we have $\binom{n}{2} = \binom{k}{2} + k(n-k) + \binom{n-k}{2}$

Proof ①: $\binom{k}{2} + k(n-k) + \binom{n-k}{2} = \frac{1}{2}k^2 - \frac{1}{2}k + nk - k^2 + \frac{1}{2}(n-k)^2 - \frac{1}{2}(n-k)$

$= \cancel{\tfrac{1}{2}k^2} - \cancel{\tfrac{1}{2}k} + \cancel{nk} - \cancel{k^2} + \frac{1}{2}n^2 \cancel{- nk} + \cancel{\tfrac{1}{2}k^2} - \frac{1}{2}n + \cancel{\tfrac{1}{2}k}$

$= \frac{1}{2}n^2 - \frac{1}{2}n = \binom{n}{2}$

② Consider a subset $U \subseteq \{1, \ldots, n\}$ with $|U| = k$. Then
$\binom{n}{2}$ = # subsets of size 2 in $\{1, \ldots, n\}$
$\binom{k}{2}$ = number with both elements in U
$k(n-k)$ = number with one element in U & one in U^c
$\binom{n-k}{2}$ = " " both elements in U^c □.

Now consider a list $s_1, \dots, s_n$ of nonnegative integers.
When can this be the list of scores for a tournament of n players?

First condition: each of the $\binom{n}{2}$ games gives a score of 1 to one player & 0 to the other $\therefore$ $s_1 + \dots + s_n$ = total score of all players = total number of games $= \binom{n}{2}$.

From now on we assume that $s_1 + \dots + s_n = \binom{n}{2}$.
For any subset $U \subseteq \{1, \dots, n\}$ we put $s_U = \sum_{i \in U} s_i$, eg $s_{\{1,4,7\}} = s_1 + s_4 + s_7$. Note that we always have $s_U + s_{U^c} = s_1 + \dots + s_n = \binom{n}{2}$.

Definition: the list s is plausible iff for all U we have $s_U \geq \binom{|U|}{2}$.

Lemma: if s is plausible then for all U we have $s_U \leq \binom{|U|}{2} + |U|(n-|U|)$
In other words, if $|U| = k$ then $s_U \leq \binom{k}{2} + k(n-k)$.

Proof: suppose that $|U| = k$ so $|U^c| = n-k$. By the plausibility condition for U^c, we have $s_{U^c} \geq \binom{n-k}{2}$. This gives

$$s_U = \binom{n}{2} - s_{U^c} \leq \binom{n}{2} - \binom{n-k}{2} = \left(\binom{k}{2} + k(n-k) + \binom{n-k}{2}\right) - \binom{n-k}{2}$$
$$= \binom{k}{2} + k(n-k) \quad \square.$$

Note that the above argument is easily reversible, so in fact s is plausible iff for all U we have $s_U \leq \binom{k}{2} + k(n-k)$.

Lemma: if the numbers s_i are the scores from a tournament, then the list s is plausible.

Proof: in this case s_U = total scores of players in U = total from games between members of U + total from games between U & U^c = $\binom{k}{2}$ + (a number between 0 & $k(n-k)$) so $\binom{k}{2} \leq s_U \leq \binom{k}{2} + k(n-k)$ $\square$.

Theorem (Landau): The converse also holds: if the list $s_1, \dots, s_n$ is plausible then there exists a tournament with scores $s_1, \dots, s_n$.

Proof: we will construct a matching problem, use the team version of Hall's Theorem to show that the matching problem can be solved, & then see that this gives the required tournament.

We make a set T of trophies, with one trophy for each game, so there are $\binom{n}{2}$ trophies. Each one has the names of the two players (but it does not say who won). Our problem is to allocate trophies in such a way that

(a) Each player only gets trophies for the games that they played in, ie trophies that have their name on.

(b) Player i gets s_i trophies in total.

This is mathematically identical to a team allocation problem with jobs $j_1, \dots, j_n$, and $\binom{n}{2}$ candidates, where the candidates for j_i correspond to the trophies with i's name on.

By the team version of Hall's theorem, the problem is solvable iff for every set U of players, the total number of trophies that could be allocated to players in U is at least $\sum_{i \in U} s_i$. If $|U| = k$, the allocatable trophies are the $\binom{k}{2}$ trophies with both names in U, together with the $k(n-k)$ trophies with one name from U & one from U^c. So the condition is $s_U \leq \binom{k}{2} + k(n-k)$, which is precisely the inequality that we assumed. $\square$.

Addendum: suppose that the numbers $s_1, \dots, s_n$ are in decreasing order. Then for any U with $|U| = k$ we have

$s_{n-k+1} + \dots + s_n \leq s_U \leq s_1 + \dots + s_k$ $\therefore$ s is plausible iff

$\forall k \;\; s_1 + \dots + s_k \leq \binom{k}{2} + k(n-k)$ iff $\forall k \;\; \underbrace{s_{n-k+1} + \dots + s_n}_{= \text{last } k \ s\text{'s}} \geq \binom{k}{2}$.

(18)

Summary: for $s_1, \ldots, s_n \in \mathbb{N}$ with $s_1 + \cdots + s_n = \binom{n}{2}$, the following are equivalent:

(1) There is a tournament of n players with scores $s_1, \ldots, s_n$

(2) The sum of any k of the s_i is at least $\binom{k}{2}$

(3) The sum of any $n-k$ of the s_i is at most $\binom{k}{2} + k(n-k)$

If $s_1 \geq s_2 \geq \ldots \geq s_n$ then the following conditions are also equivalent

(4) The sum of the last k of the s_i is at least $\binom{k}{2}$

(5) " " " " first k of the s_i is at most $\binom{k}{2} + k(n-k)$.

Eg $s = 5\ 3\ 2\ 2\ 2\ 1$, $n = 6$, $s_1 + \cdots + s_6 = 15 = \binom{6}{2}$ ✓.

$1 \geq \binom{1}{2} = 0$ ✓

$2+1 \geq \binom{2}{2} = 1$ ✓

$2+2+1 \geq \binom{3}{2} = 3$ ✓

$2+2+2+1 \geq \binom{4}{2} = 6$ ✓

$3+2+2+2+1 \geq \binom{5}{2} = 10$ ✓

$5 + \cdots + 1 = \binom{6}{2} = 15$ ✓

	1	2	3	4	5	6
1		W	W	W	W	W
2	L		L	W	W	W
3	L	W		L	L	W
4	L	L	W		L	W
5	L	L	W	W		L
6	L	L	L	L	W	

The table on the left shows that the plausibility conditions are satisfied so there exists a tournament with these scores. The table on the right gives an example of such a tournament.

Eg: $s = 4\,4\,4\,2\,1\,1\,1$, sum of last 4 is $5 < \binom{4}{2}$ $\therefore$ not plausible, so this cannot be the list of scores from a tournament.

Eg: $s = 5\,4\,4\,1\,1\,0$, sum of last 3 is $2 < \binom{3}{2}$ $\therefore$ not plausible.

Eg: $s = 9\,8\,7\,6\,5\,4\,3\,2\,1$: these are scores from a tournament in which the lower numbered player always wins, ie player 1 is consistently better than player 2, player 2 is consistently better than player 3 etc.

Eg: for an odd number $n = 2m+1$, we can imagine a tournament in which i beats j iff $j - i \in \{1, \dots, m\} \bmod 2m+1$

Eg $m=3$:

	1	2	3	4	5	6	7
1	–	W	W	W	L	L	L
2	L	–	W	W	W	L	L
3	L	L	–	W	W	W	L
4	L	L	L	–	W	W	W
5	W	L	L	L	–	W	W
6	W	W	L	L	L	–	W
7	W	W	W	L	L	L	–

This has scores $m, m, \dots, m$ (repeated n times)
Because this is the score sequence from a tournament, it must satisfy the plausibility condition. We can see this more directly:

$$\left(\begin{array}{c}\text{sum of any } k \text{ scores} \\ (\text{with } k \le n)\end{array}\right) = km = k\frac{n-1}{2} \geq k\frac{k-1}{2} = \binom{k}{2}$$

Problem: construct a tournament with scores 7777722222

Solution: Construct a tournament of players $1, \dots, 5$ with scores 22222 by case $m=2$ of previous example. Construct a similar tournament for players $6, \dots, 10$ with scores 22222. Then consider a combined tournament with players $1, \dots, 10$ in which $1, \dots, 5$ always beat $6, \dots, 10$; the scores are then 7777722222. $\square$.