MAS334 Combinatorics
Autumn Semester 2018-2019

Lecture Notes and Example Sheets

Contents

CHAPTER 1

The Binomial Coefficients

Example 1. If the 100 people here in this lecture theatre enter an Olympics marathon, in how many ways can the 3 medals be awarded?

Example 2. Suppose the 100 people here are participants in a race. Clare Balding wants to interview three of the participants. In how many ways can they be chosen?

Definition 3. The *binomial coefficient* $\binom{n}{k}$, (pronounced 'n choose k'), is defined by

$$\binom{n}{k} = \begin{cases} \frac{n(n-1)(n-2)\dots(n-k+1)}{k(k-1)(k-2)\dots 1} = \frac{n!}{k!(n-k)!}, & \text{if } 0 \le k \le n \text{ and } n,\, k \text{ are integers,} \\ 0, & \text{otherwise.} \end{cases}$$

The number of ways of choosing k items from n is $\binom{n}{k}$.

Of course, $n! = n(n-1)(n-2)\dots 2.1$ for each integer $n \geq 1$, and by convention $0! = 1$. So $\binom{n}{0} = 1$ and $\binom{n}{1} = n$ for all non-negative integers n.

Example 4. How many lottery combinations are there?

Example 5. How many x^3s are there in the expansion of $(1+x)^7$?

Generalising Example 5 leads us to the following result.

Theorem 6 (The Binomial Theorem).

$$\begin{aligned} (1+x)^n &= \binom{n}{0} + \binom{n}{1} x + \cdots + \binom{n}{k} x^k + \cdots + \binom{n}{n} x^n \\ &= \sum_{k=0}^{n} \binom{n}{k} x^k. \end{aligned}$$

We can draw the (non-zero) binomial coefficients $\binom{n}{k}$ in **Pascal's Triangle**:

n							
Row 0	1						
Row 1	1	1					
Row 2	1	2	1				
Row 3	1	3	3	1			
Row 4	1	4	6	4	1		
Row 5	1	5	$\boxed{10}$	10 ↘	↙ 5	1	
Row 6	1	6	15	20	15	6	1
	↗ $k=0$	↗ $k=1$	↗ $k=2$	…			

For example, in Row 5, $k = 2$ gives

$$\binom{5}{2} = \frac{5.4}{2.1} = 10$$

and this is the $\boxed{10}$ indicated in the triangle.
Notice also that every entry is the sum of the two above it. The example

$$\binom{5}{3} + \binom{5}{4} = 10 + 5 = 15 = \binom{6}{4}$$

is indicated with arrows in the diagram.

The general statement of this property is:

PROPOSITION 7 (Pascal's Identity). *For $n \neq 0$ and all k,*

$$\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}.$$

There is an obvious line of symmetry down the middle of Pascal's Triangle. For example,

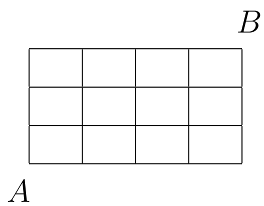
	1	5	10 |	10	5	1	
1	6	15	2|0	15	6	1	

So we have, for example, $\binom{6}{1} = \binom{6}{5}$. Here is the general statement.

PROPOSITION 8. *For all n and k,*

$$\binom{n}{k} = \binom{n}{n-k}.$$

EXAMPLE 9. Imagine that the following diagram is a grid of roads.



We want to get from A to B by as short a route as possible. How many such routes are there?

EXAMPLE 10. How many solutions are there of the equation

$$x_1 + x_2 + x_3 + x_4 = 6,$$

where each x_i is an integer and each $x_i \geq 0$.

Generalising Example 10 leads to:

PROPOSITION 11. *The number of solutions involving non-negative integers x_i of the equation*

$$x_1 + x_2 + \cdots + x_k = n$$

is

$$\binom{n+k-1}{k-1}.$$

EXAMPLE 12. How many solutions are there of the equation

$$y_1 + y_2 + \cdots + y_k = n,$$

where each y_i is an integer strictly greater than 0? (So 0 is no longer allowed.)

PROPOSITION 13. *The number of solutions involving positive integers y_i of the equation*

$$y_1 + y_2 + \cdots + y_k = n$$

is

$$\binom{n-1}{k-1}.$$

EXAMPLE 14. There are n seats in a row in a doctor's waiting room. There are k patients who want to choose seats with no two adjacent. In how many ways can the k seats be chosen?

EXAMPLE 15. Of the 45 057 474 lottery combinations, what proportion have at least 2 of their numbers consecutive?

EXAMPLE 16. Let m and n be integers with $1 \le m \le n$. By considering choosing m members from $\{1, 2, 3, \ldots, n\}$ and by looking at the highest choice, show that

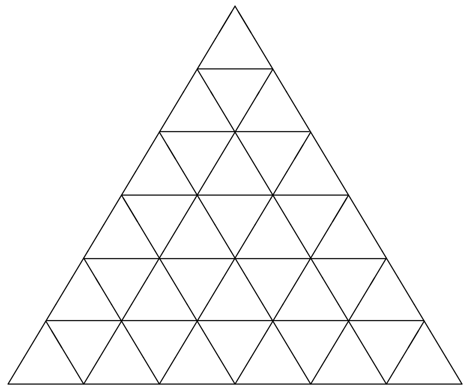
$$\binom{n}{m} = \binom{n-1}{m-1} + \binom{n-2}{m-1} + \binom{n-3}{m-1} + \cdots + \binom{m-1}{m-1} = \sum_{k=m}^{n} \binom{k-1}{m-1}.$$

EXAMPLE 17. What is the total of all these numbers?

$$\begin{array}{ccccccccccccc}
 & & & & & & 1 & & & & & & \\
 & & & & & 1 & & 2 & & & & & \\
 & & & & 1 & & 2 & & 3 & & & & \\
 & & & 1 & & 2 & & 3 & & 4 & & & \\
 & & . & & . & & . & & . & & . & & \\
 & . & & . & & . & & . & & . & & . & \\
1 & & 2 & & 3 & & 4 & & \ldots & & \ldots & & N
\end{array}$$

Express your answer as a single binomial coefficient.

EXAMPLE 18. How many triangles can be seen pointing upwards in this diagram?



How many when there are N rows?

EXAMPLE 19. The *Fibonacci sequence* $(f_n)_{n\geq 1}$ is defined by

$$f_1 = 1, \quad f_2 = 2,$$
$$f_{n+2} = f_{n+1} + f_n, \qquad \text{for all } n \geq 1.$$

The first few terms of the sequence are:

$$1, 2, 3, 5, 8, 13, 21, 34, 55, 89, ...$$

Show that

$$f_n = \binom{n}{0} + \binom{n-1}{1} + \binom{n-2}{2} + \cdots$$

(Here the dots indicate that we continue the terms in the sum until they become zero. Note that we can also write

$$\sum_k \binom{n-k}{k}$$

for this sum. There is no need to specify the range of values of k, since we want all possible values of k which contribute non-zero terms to the sum.)

EXAMPLE 20. We mark n different points on the circumference of a circle. Each pair of points is joined by a straight line. This is done in such a way that no three of the lines meet at a point inside the circle.

(1) How many lines are there?
(2) How many internal crossing points are there?
(3) How many regions are there?

CHAPTER 2

Three Basic Principles

1. Parity

Example 21. How many solutions are there of the equation

$$2x + 6y = 11 \ ?$$

In how many of these solutions are both x and y integers?

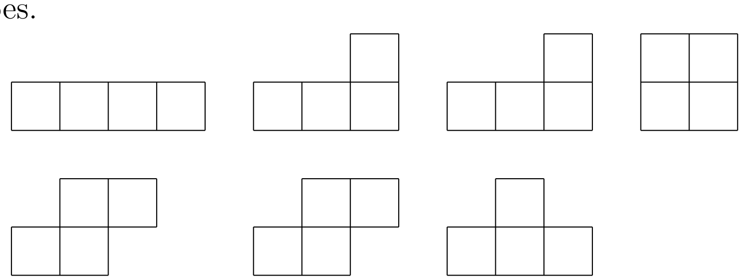
Note. You can prove that a situation is **impossible** by a parity mis-match. You *cannot* prove that something is possible by just checking the parity match.

For example, are there integer solutions of $12x+18y = 250$? No, because the LHS is divisible by 3 and the RHS is not. So there is no solution despite the odd/even parity match.

Example 22. You are given an $n \times n$ chessboard and some dominoes each of which can cover two adjacent squares of the board.

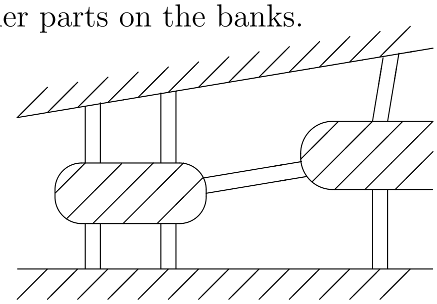
(1) Show that the board can be covered completely with non-overlapping dominoes if and only if n is even.
(2) Show that if two squares are removed from opposite corners of the board then the remaining board cannot be covered with non-overlapping dominoes.

Example 23. A nice present I got one Christmas was a light based on the game tetris. It has seven components, each of which is made up of four squares, in the following shapes.



The light can be assembled in many ways, with each piece lighting up as it is added. At New Year a friend spent some time attempting to arrange the pieces into a four by seven rectangle. Show that this is impossible.

EXAMPLE 24. Here is a schematic plan of the town of Königsburg. A river runs through the town and the parts of the town on two islands in the river are joined by bridges to the other parts on the banks.



In 1736 inhabitants of the town considered the following problem: is it possible to walk around the town using each bridge once and only once? Euler proved it was impossible by a parity argument.

2. The Pigeon-Hole Principle

The Pigeon-Hole Principle says that if you place more than n letters in n pigeon-holes then some pigeon-hole will contain more than one letter.

This is completely obvious, but it has some powerful consequences.

EXAMPLE 25. Amongst a group of people some handshaking takes place (no-one shakes their own hand, no pair shake more than once). Show that there are two people who shake the same (positive) number of hands.

EXAMPLE 26. There are (at least) two people in the world with the same number of hairs.

EXAMPLE 27. Given any 10 different positive integers less than 100, there will be two disjoint subsets with the same sum.

EXAMPLE 28. Show that, given any sequence of n integers,

$$x_1, x_2, x_3, \ldots, x_n$$

some consecutive collection has a sum divisible by n.

EXAMPLE 29. Each day for 100 days I put either £1 or £2 into a piggy-bank. On 50 days it's £1 and on the other 50 days it's £2. Let k be an integer with $1 \leq k < 50$. Show that over some consecutive period of days I will put a total of precisely £k into the piggy-bank.

3. Inclusion/Exclusion

EXAMPLE 30. In a sports club

10	people play tennis,
12	people play squash,
3	people play both tennis and squash.

How many people in the club play at least one of these two sports?

EXAMPLE 31. In a sports club

10	people play tennis,
15	play squash,
12	play badminton,
5	play tennis and squash,
4	play tennis and badminton,
3	play squash and badminton,
2	play all three sports.

How many play at least one of these three sports?

THEOREM 32 (The Principle of Inclusion/Exclusion). *Suppose we have a finite set of items and properties* $1, 2, 3, \ldots, n$*. Let* $N(i_1, i_2, \ldots, i_r)$ *be the number of items which have the properties* $i_1, i_2, \ldots, i_r$ *(and maybe others). Then the number of items with at least one of the properties is*

$$\begin{aligned}
&N(1) + N(2) + N(3) + \cdots + N(n) \\
&- N(1,2) - N(1,3) - \cdots - N(n-1,n) \\
&+ N(1,2,3) + N(1,2,4) + \cdots \\
&- N(1,2,3,4) - \cdots \\
&\qquad\vdots \\
&+ (-1)^{n-1} N(1,2,3,\ldots,n).
\end{aligned}$$

This sum may also be written

$$\sum_{(i_1, i_2, \ldots, i_r)} (-1)^{r-1} N(i_1, i_2, \ldots, i_r).$$

EXAMPLE 33. A permutation of $\{1, 2, \ldots, n\}$ is called a *derangement* if $1 \not\mapsto 1$, $2 \not\mapsto 2$, ..., $n \not\mapsto n$, i.e. no number is mapped to itself.

(1) How many derangements are there of $\{1, 2, 3, \ldots, n\}$?
(2) Show that the probability of a permutation being a derangement tends to $1/e$ as n tends to ∞.

EXAMPLE 34. List the numbers in $\{1, 2, \ldots, 42\}$ which are relatively prime to 42. How many are there?

THEOREM 35 (Euler's Function). *Let m be a positive integer whose distinct prime factors are p_1, p_2, p_3, ..., p_n. Then the number of integers from $\{1, 2, 3, \ldots, m\}$ which are relatively prime to m is*

$$m\left(1-\frac{1}{p_1}\right)\left(1-\frac{1}{p_2}\right)\cdots\left(1-\frac{1}{p_n}\right).$$

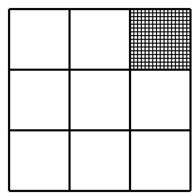
CHAPTER 3

Rook Polynomials

1. Definition of Rook Polynomials

In chess a rook (also called a castle) challenges another piece if it is in the same row or column.

EXAMPLE 36. Look at the unshaded board in the diagram.



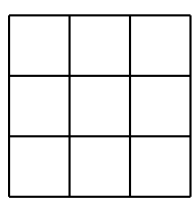
(1) In how many ways can 1 rook be placed on the board?
(2) In how many ways can 2 rooks be placed on the board so that neither challenges the other (i.e. in different rows and columns)?
(3) What about 3 rooks?

DEFINITION 37. If B is part of an $n \times n$ board, the *rook polynomial* of the board B is

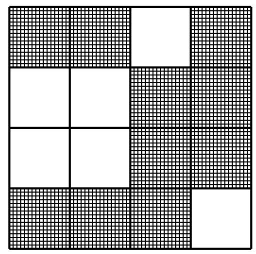
$$r_B(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_k x^k + \cdots + a_n x^n,$$

where the coefficient a_k is the number of ways k rooks can be placed on the board B without challenging each other. Note that $a_0 = 1$, since there is just one way of placing zero rooks.

EXAMPLES 38. (Rook polynomials calculated by bare hands.)



$1 + 9x + 18x^2 + 6x^3$



$$1 + 6x + 11x^2 + 8x^3 + 2x^4$$

This is worth doing because

(1) lots of problems reduce to rook-type problems,
(2) there is a standard algorithm for calculating rook polynomials.

2. Some Problems which Reduce to Rooks

Example 39. How many permutations of $\{1, 2, 3, 4, 5\}$ are there such that

$$1 \not\mapsto 1\,, 1 \not\mapsto 2\,, 2 \not\mapsto 2\,, 2 \not\mapsto 3\,, 3 \not\mapsto 3\,, 3 \not\mapsto 4\,, 4 \not\mapsto 4\,, 4 \not\mapsto 5\,, 5 \not\mapsto 5\,?$$

Example 40. Four people A, B, C, D are to be allocated a job each from jobs a, b, c, d, subject to the following conditions.

A cannot do b or c,
B cannot do a,
C cannot do a, b or d,
D cannot do c or d.

In how many ways can the 4 jobs be allocated?

Example 41 (Hostess Problem or Problème des Ménages). Five heterosexual couples want to sit round a circular table, alternating man, woman, man, woman, ... and such that no woman is sitting next to her partner. In how many ways can it be done?

Example 42 (Snap Problem). We have two full packs of cards. The second pack is in order from top to bottom: AC, AH, AD, AS, KC, KH, KD, KS, ..., 2C, 2H, 2D, 2S. We go through the packs comparing corresponding cards. If two have the same value (not necessarily the same suit) it's a 'snap'. How many different orderings of the first pack lead to no snaps?

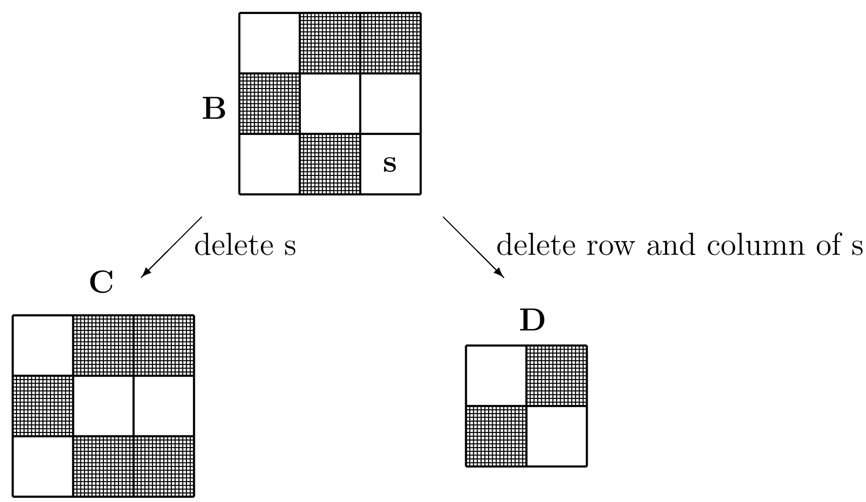
3. Calculating Rook Polynomials

Recall that we write $r_B(x)$ for the rook polynomial of a board B.

THEOREM 43. *Let B be part of an $n \times n$ board and let s be one specified square of B. Let C be the board B with s deleted and let D be B with the whole of s's row and column deleted. Then*

$$r_B(x) = r_C(x) + xr_D(x).$$

EXAMPLE 44. We illustrate the theorem for a particular board B and square s.

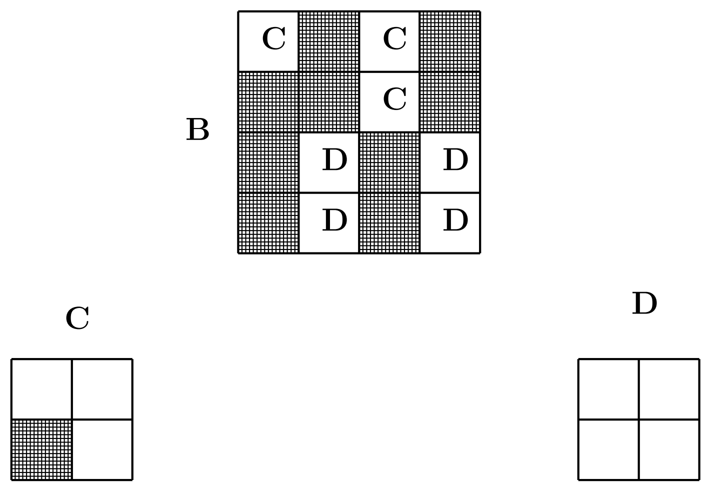


Here

$$r_C(x) = 1 + 4x + 4x^2,$$
$$r_D(x) = 1 + 2x + x^2,$$
$$r_B(x) = 1 + 5x + 6x^2 + x^3 = r_C(x) + xr_D(x).$$

Another shortcut works in the situation when a board B can be split into two pieces, C and D, so that the two pieces do not share any row or column. We write $B = C \cup D$ in this case.

Example 45. We have $B = C \cup D$, where B, C and D are as in the diagram.



Theorem 46. *Let B be part of an $n \times n$ board and suppose that $B = C \cup D$. Then*

$$r_B(x) = r_C(x) r_D(x).$$

Example 47. Use Theorems 43 and 46 to calculate the rook polynomial of the job allocation board from Example 40.

Example 48. Use rook polynomials to find the number of ways of adding a 4th row of the numbers 1 to 5 to

1	2	3	4	5
2	4	5	3	1
4	3	1	5	2

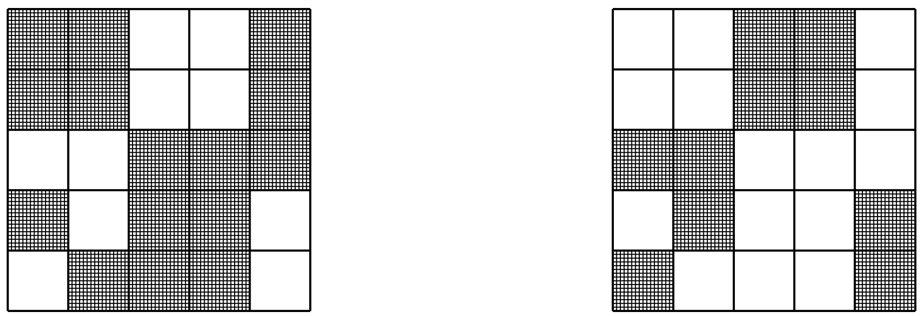
so that there are no repeats in any column.

Next we explain another way to get information about the rook polynomial of one board from the polynomial of a related board.

Definition 49. If B is the unshaded part of an $n \times n$ board, then the *complement* of B, written $\overline{B}$, is given by the shaded part of the $n \times n$ board.

Example 50. A board B and its complement $\overline{B}$.

B $\overline{\mathbf{B}}$



The following theorem allows us to calculate the highest coefficient of the rook polynomial of $\overline{B}$ from the rook polynomial of B.

Theorem 51. *Let B be part of an $n \times n$ board and let its rook polynomial be*

$$1 + r_1 x + r_2 x^2 + \cdots + r_n x^n.$$

Then the number of ways of placing n non-challenging rooks on $\overline{B}$, the complement of B, is

$$\sum_{k=0}^{n} (-1)^k (n-k)! r_k,$$

where $r_0 = 1$.

Example 52. In Example 50, we have

$$r_B(x) = 1 + 10x + 35x^2 + 50x^3 + 26x^4 + 4x^5,$$
$$r_{\overline{B}}(x) = 1 + 15x + 75x^2 + 145x^3 + 96x^4 + 12x^5.$$

Note that

$$(5! \times 1) - (4! \times 10) + (3! \times 35) - (2! \times 50) + (1! \times 26) - (0! \times 4) = 12.$$

We will end this section by going back to Examples 41 and 42, the Hostess and Snap Problems. Earlier we showed how to reduce these to rook problems. Now we give complete solutions.

CHAPTER 4

Hall's Marriage Theorem

This chapter covers an important result about when you can select different elements from each of a collection of sets. There is a traditional formulation in terms of marriage which we begin with. All sets considered in this chapter will be finite.

1. The Marriage Theorem

EXAMPLE 53. In a group of 7 men and 6 women

woman 1	knows men	$1', 2', 3'$,
woman 2	knows men	$2', 3'$,
woman 3	knows men	$3', 5', 7'$,
woman 4	knows men	$1', 2'$,
woman 5	knows men	$1', 2', 3'$,
woman 6	knows men	$4', 5', 6'$.

Can each woman find a husband from the men she knows?

THEOREM 54 (Hall's Theorem - Marriage Version).

A finite set of women can always find husbands from amongst the men they know. $\iff$ *For each r, any set of r of the women know at least r men between them.*

COROLLARY 55. *Let $d > 0$ be fixed. Consider a finite group of men and women where each woman knows at least d men and each man knows at most d women. Then each woman can find a husband.*

COROLLARY 56. *In the same situation as Corollary 55, if P is the set of men who know* exactly *d women, then the husbands can be chosen to include P.*

Corollary 57 (Harem Version). *Consider a society in which a woman may have many husbands, but a man may have only one wife. In this situation, consider a group of* n *women and suppose that the* i*th woman wants to choose* m_i *husbands from amongst the men she knows. This is possible if and only if any set of the women,* $(i_1, i_2, \ldots, i_r)$*, say, know between them at least* $m_{i_1} + m_{i_2} + \cdots + m_{i_r}$ *men.*

Example 58. The sets

$$\begin{aligned} A_1 &= \{\mathbf{1}, 3\}, \\ A_2 &= \{2, \mathbf{3}\}, \\ A_3 &= \{1, 3, \mathbf{4}, 5\}, \\ A_4 &= \{2, 4, \mathbf{6}\}, \\ A_5 &= \{1, \mathbf{5}\}, \\ A_6 &= \{1, \mathbf{2}\}, \end{aligned}$$

have *distinct representatives*, (also known as a *transversal*), for example those elements indicated in bold. However, the sets

$$\begin{aligned} A_1 &= \{1, 2, 3\}, \\ A_2 &= \{2, 3\}, \\ A_3 &= \{3, 5, 7\}, \\ A_4 &= \{1, 2\}, \\ A_5 &= \{1, 2, 3\}, \\ A_6 &= \{4, 5, 6\}, \end{aligned}$$

do not have distinct representatives (since, for example, the four sets A_1, A_2, A_4, A_5 contain between them only three members 1, 2, 3).

Corollary 59 (Hall's Theorem - Transversal Version).

Finite sets $A_1, A_2, \ldots A_n$ *have distinct representatives.* $\iff$ *For each* r*, any collection of* r *of the sets contains at least* r *elements.*

$$\iff \left| \bigcup_{i \in I} A_i \right| \geq |I| \quad \textit{for all } I \subseteq \{1, 2, \ldots, n\}.$$

Corollary 60. *Let* A_1*,* A_2*, ...,* A_n *be subsets of* $\{1, 2, \ldots, n\}$ *with*

$$|\,A_1| = |\,A_2| = \cdots = |\,A_n| = d > 0$$

and such that for $1 \leq i \leq n$*, each* i *is in precisely* d *of the sets. Then* A_1*,* A_2*, ...,* A_n *have distinct representatives.*

COROLLARY 61 (Harem Transversal Version). *Given finite sets A_1, A_2, ..., A_n, the following conditions are equivalent.*

(1) *We can find, for $1 \le i \le n$, w_i representatives of A_i, (with the $w_1, w_2, \ldots, w_n$ all different representatives).*
(2) *Given any collection of the sets, $A_{i_1}, A_{i_2}, \ldots, A_{i_r}$, say, they contain between them at least $w_{i_1} + w_{i_2} + \cdots + w_{i_r}$ elements.*
(3) $\left|\bigcup_{i\in I} A_i\right| \ge \sum_{i\in I} w_i$ *for all $I \subseteq \{1, 2, \ldots, n\}$.*

EXAMPLE 62. In the matrix

$$\begin{pmatrix} 5 & 3 & 0 & \mathbf{0} \\ \mathbf{0} & 1 & 1 & 0 \\ 2 & 4 & \mathbf{0} & 1 \end{pmatrix}$$

there is a zero in every row with no pair of these zeroes in the same column, as shown in bold for example.

COROLLARY 63 (Hall's Theorem - Matrix Version). *Let M be a matrix. The following conditions are equivalent.*

(1) *There exists a 0 in each row of M with no pair of these zeroes in the same column.*
(2) *For any r rows of M, those rows between them contain 0s in at least r columns.*

2. Tournaments

DEFINITION 64. A *tournament* of n players consists of $\binom{n}{2}$ games where each player plays each of the others once, each game resulting in a win for one of the players.

EXAMPLE 65. Players A, B, C, D, E play in a tournament. Here are the games played, with the winners in bold:

$$\mathbf{A}B, \quad \mathbf{A}C, \quad A\mathbf{D}, \quad \mathbf{A}E, \quad B\mathbf{C}, \quad \mathbf{B}D, \quad \mathbf{B}E, \quad \mathbf{C}D, \quad C\mathbf{E}, \quad \mathbf{D}E.$$

Thus A won three games, B, C, D all won two and E only won one. We say that the scores are: $3, 2, 2, 2, 1$. Note that

$$A \quad \text{beat} \quad B \quad \text{beat} \quad D \quad \text{beat} \quad E \quad \text{beat} \quad C.$$

THEOREM 66. *In any tournament of n players, they can be put in order $p_1, \ldots, p_n$, so that*

$$p_1 \quad \textit{beat} \quad p_2 \quad \textit{beat} \quad p_3 \quad \textit{beat} \quad \ldots \quad \textit{beat} \quad p_n.$$

EXAMPLE 67. Which of the following are possible sets of scores for a tournament of 6 players?

(1) 5, 3, 2, 2, 2, 1.
(2) 4, 4, 4, 2, 1, 1.
(3) 5, 4, 4, 1, 1, 0.

THEOREM 68 (Landau). *Let $w_1, w_2, w_3, \ldots, w_n$ be non-negative integers with $w_1 + w_2 + w_3 + \cdots + w_n = \binom{n}{2}$. Then the following conditions are equivalent.*

(1) *There is a tournament with scores $w_1, w_2, w_3, \ldots, w_n$.*
(2) *Any r of the w_is add to at least $\binom{r}{2}$.*
(3) *Any r of the w_is add to at most $(n-r) + \cdots + (n-2) + (n-1)$.*

Note that $(n-r) + \cdots + (n-2) + (n-1) = rn - \binom{r+1}{2}$.

3. An Application to Matrices and Job Grids

In a matrix we will use the word *line* to mean a row or a column.

EXAMPLE 69. In the matrix

$$\begin{pmatrix} 0 & 1 & 2 & 3 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 \\ 0 & 2 & 2 & 2 \end{pmatrix}$$

(1) how many 0s can you see with no pair of these zeroes on the same line?
(2) what is the minimum number of lines which include all the 0s?

THEOREM 70 (König-Egerváry). *In any matrix:*

The minimum number of lines to include all the $0s$	$=$	*the maximum number of* $0s$ *with no pair of these zeroes on the same line.*

EXAMPLE 71 (Hungarian Algorithm). (This example requires the algorithm only once. In general the algorithm increases the number of non-challenging 0s by 1 and it may have to be reapplied several times.)

Four people A, B, C, D have to be allocated one job each from a, b, c, d. The table shows their 'unsuitability' for the jobs - the *lower* numbers meaning they are *better* suited for the job. (Think of 'time taken' to complete the task.)

	a	b	c	d
A	1	0	1	0
B	1	3	0	2
C	0	4	0	2
D	0	3	1	4

Allocate the jobs so that total unsuitability is least.

CHAPTER 5

Latin Squares

1. Extending Latin Rectangles

EXAMPLES 72.

$$\begin{pmatrix} 1 & 4 & 3 & 2 \\ 2 & 3 & 4 & 1 \\ 4 & 1 & 2 & 3 \\ 3 & 2 & 1 & 4 \end{pmatrix} \qquad \text{is a } 4 \times 4 \text{ Latin square.}$$

$$\begin{pmatrix} 1 & 4 & 3 \\ 5 & 2 & 1 \end{pmatrix} \qquad \text{is a } 2 \times 3 \text{ Latin rectangle.}$$

DEFINITION 73. An $n \times n$ *Latin square* is an $n \times n$ matrix in which every one of the numbers $1, 2, \ldots, n$ appears in each row and each column.
A $p \times q$ *Latin rectangle* with entries in $\{1, 2, \ldots, n\}$ is a $p \times q$ matrix whose entries are from $\{1, 2, \ldots, n\}$ with no repeat in any row or column.

NOTE. An $n \times n$ Latin rectangle with entries in $\{1, 2, \ldots, n\}$ is an $n \times n$ Latin square.

REMARK 74. A correctly completed sudoku grid is a 9×9 Latin square satisfying the extra condition that each of the numbers $1, \ldots, 9$ appears exactly once in each 3×3 box. Their study is not part of this course, but if you are interested, see Dr. Frazer Jarvis' sudoku pages at www.afjarvis.staff.shef.ac.uk/sudoku.

EXAMPLE 75. Can the Latin rectangle $\begin{pmatrix} 1 & 2 & 4 & 5 & 3 \\ 5 & 1 & 2 & 3 & 4 \end{pmatrix}$ be extended to a 5×5 Latin square?

For convenience, we recall here a result (Corollary 60) from the previous chapter. We will use it in the proof of the next theorem.

Let A_1, A_2, ..., A_n be subsets of $\{1, 2, \ldots, n\}$ with

$$|A_1| = |A_2| = \cdots = |A_n| = d > 0$$

and such that for $1 \leq i \leq n$, each i is in precisely d of the sets. Then A_1, A_2, ..., A_n have distinct representatives.

THEOREM 76. *Let $p < n$ and let L be a $p \times n$ Latin rectangle with entries in $\{1, 2, \dots, n\}$. Then L can be extended to an $n \times n$ Latin square.*

EXAMPLE 77. Show that $\begin{pmatrix} 6 & 1 & 2 & 3 \\ 5 & 6 & 3 & 1 \\ 1 & 3 & 6 & 2 \\ 3 & 2 & 4 & 6 \end{pmatrix}$ cannot be extended to a 6×6 Latin square.

So, although $p \times \underline{n}$ rectangles can be extended to $n \times n$ squares, in general $p \times q$ rectangles cannot.

In a sense, the extension failed in the previous example because 5 does not appear enough times in the original rectangle. We make precise the condition needed in the next theorem. Again the proof will need a result (Corollary 56) from the previous chapter. The following lemma just restates this result in different language.

LEMMA 78. *Let $A_1, A_2, \dots, A_p$ be subsets of $\{1, 2, \dots, n\}$ and let*

$$P = \{i \mid 1 \le i \le n \text{ and } i \text{ is in precisely } d \text{ of the sets}\}.$$

If

(1) $|A_1| = |A_2| = \cdots = |A_p| = d\ (> 0)$, *and*
(2) *for $1 \le i \le n$, each i is in at most d of the sets,*

then $A_1, A_2, \dots, A_p$ have distinct representatives and those representatives can be chosen to include P.

THEOREM 79. *Let L be a $p \times q$ Latin rectangle with entries in $\{1, 2, \dots, n\}$. For $1 \le i \le n$, let $L(i)$ be the number of occurrences of i in L. Then L can be extended to an $n \times n$ Latin square if and only if $L(i) \ge p + q - n$ for each i.*

EXAMPLE 80. Use the process described in the proof of Theorem 79 to extend

$$\begin{pmatrix} 5 & 6 & 1 & 4 \\ 6 & 5 & 4 & 7 \\ 1 & 2 & 3 & 5 \\ 3 & 4 & 5 & 6 \\ 2 & 7 & 6 & 1 \end{pmatrix}$$

to a 7×7 Latin square.

2. Orthogonal Latin Squares

EXAMPLE 81 (Sixteen Officers). There are 16 soldiers, 4 from each regiment $1, 2, 3, 4$ and in each regiment they are of rank $1, 2, 3, 4$. Write (i, j) to denote the soldier in regiment i of rank j. Arrange the soldiers in a 4×4 array so that each row and each column contains a member of each regiment <u>and</u> with one of each rank in every row and column.

DEFINITION 82. Two $n \times n$ Latin squares $L = (l_{ij}), M = (m_{ij})$ are called *orthogonal* if the pairs (l_{ij}, m_{ij}) include all the possibilities $(1, 1), (1, 2), \ldots, (n, n)$.

When is it possible to construct two such orthogonal Latin squares? We have seen that we can do it for $n = 4$. On the other hand, it is easy to see that it is impossible for $n = 2$.

In 1782 Euler considered the 6×6 case of thirty-six officers. He conjectured that it was impossible to construct two orthogonal Latin squares in this case. It took over 100 years before this was proved, by Tarry in 1900.

After Euler's work it was also conjectured that it would be impossible for all n of the form $4m + 2$, but this turned out to be wrong. It took until around 1960 before it was shown that in fact, $n = 2$ and $n = 6$ are the *only* impossible cases.

THEOREM 83. *If n is a prime (or a power of a prime) then there exist $n - 1$ $n \times n$ Latin squares, $L_1, \ldots, L_{n-1}$, which are mutually orthogonal (i.e. each pair is orthogonal).*

EXAMPLE 84. Here's how it works for $n = 5$; four mutually orthogonal 5×5 Latin squares are shown. The same process works for any prime. You should think about why this fails for composite numbers.

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 3 & 4 & 5 & 1 \\ 3 & 4 & 5 & 1 & 2 \\ 4 & 5 & 1 & 2 & 3 \\ 5 & 1 & 2 & 3 & 4 \end{pmatrix} \quad \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 3 & 4 & 5 & 1 & 2 \\ 5 & 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 5 & 1 \\ 4 & 5 & 1 & 2 & 3 \end{pmatrix} \quad \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 4 & 5 & 1 & 2 & 3 \\ 2 & 3 & 4 & 5 & 1 \\ 5 & 1 & 2 & 3 & 4 \\ 3 & 4 & 5 & 1 & 2 \end{pmatrix} \quad \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 5 & 1 & 2 & 3 & 4 \\ 4 & 5 & 1 & 2 & 3 \\ 3 & 4 & 5 & 1 & 2 \\ 2 & 3 & 4 & 5 & 1 \end{pmatrix}$$

Each row shifts ...

by 1   by 2   by 3   by 4.

THEOREM 85. *There are at most $n-1$ mutually orthogonal $n \times n$ Latin squares.*

CONJECTURE 86. There exist $n-1$ mutually orthogonal $n \times n$ Latin squares if and only if n is a prime or a power of a prime.

Progress so far on this conjecture is shown in the table.

n	n prime power?	$n-1$ orthog. Latin squares exist?	Comment
3	Y	Y	By Theorem 83.
4	Y	Y	By Theorem 83.
5	Y	Y	By Theorem 83.
$6 = 2.3$	N	N	Conjectured by Euler that there aren't 2 orthogonal squares, proved by Tarry in 1900.
7	Y	Y	By Theorem 83.
$8 = 2^3$	Y	Y	By Theorem 83.
$9 = 3^2$	Y	Y	By Theorem 83.
$10 = 2.5$	N	N	Proved in the 1980's. There *are* 2 orthogonal squares, it's not known if there are 3. Several workers in the field conjecture there are exactly 2.
11	Y	Y	By Theorem 83.
$12 = 2^2.3$	N	Not known.	It is known that 5 exist, but not if 6 do.

In case you are tempted to start tackling the $n = 12$ case, it is perhaps worth noting that the total number of 12×12 Latin squares is of the order of 10^{60}.

CHAPTER 6

Designs and Codes

1. Fair Experiments

A *design* is a mathematical object invented with a view to designing fair experiments.

EXAMPLE 87. Nine types of coffee are to be tested. Each of twelve families is asked to compare three of the types. Overall we want each pair from the nine to be compared by the same number of families. (In fact, in this example, we'll ask for each to be compared by just one family). Here is one way to do it.

Family	'Block' of varieties to be tested
1	$\{1, 2, 3\}$
2	$\{4, 5, 6\}$
3	$\{7, 8, 9\}$
4	$\{1, 4, 7\}$
5	$\{1, 5, 9\}$
6	$\{2, 5, 8\}$
7	$\{3, 6, 9\}$
8	$\{2, 6, 7\}$
9	$\{3, 4, 8\}$
10	$\{1, 6, 8\}$
11	$\{2, 4, 9\}$
12	$\{3, 5, 7\}$

DEFINITION 88. A *design* consists of v varieties and b blocks, each block consisting of k varieties and with each pair of varieties in precisely λ blocks. It will follow, by the next theorem, that each variety is then in the same number of blocks, called r. The design is then called a (v, b, r, k, λ) *design*.

We will assume throughout our work on designs that $1 < k < v$. (The cases $k = 1$ and $k = v$ are trivial.)

THEOREM 89. *Given a design of v varieties, b blocks, k varieties per block and every pair of varieties appearing in λ blocks, then each variety is in r blocks, where*

$$r = \frac{bk}{v} = \frac{\lambda(v-1)}{k-1}.$$

In the coffee example, $\frac{bk}{v} = \frac{12.3}{9} = 4$ and $\frac{\lambda(v-1)}{k-1} = \frac{1.(9-1)}{3-1} = 4$. This is a $(9, 12, 4, 3, 1)$ design.

NOTE. Not every collection of numbers satisfying the theorem necessarily corresponds to a design.

Designs are hard to construct, but sometimes modular arithmetic works.

EXAMPLE 90. We use arithmetic mod 11 to construct a design with 11 varieties and 5 varieties per block, $v = 11, k = 5$. The blocks are listed below. Each one is constructed from the previous one by adding one to the entries and working mod 11 (with the slight variation that we write 11 rather than 0).

block number	block
1	$\{1, 3, 4, 5, 9\}$
2	$\{2, 4, 5, 6, 10\}$
3	$\{3, 5, 6, 7, 11\}$
4	$\{4, 6, 7, 8, 1\} = \{1, 4, 6, 7, 8\}$
5	$\{2, 5, 7, 8, 9\}$
6	$\{3, 6, 8, 9, 10\}$
7	$\{4, 7, 9, 10, 11\}$
8	$\{1, 5, 8, 10, 11\}$
9	$\{1, 2, 6, 9, 11\}$
10	$\{1, 2, 3, 7, 10\}$
11	$\{2, 3, 4, 8, 11\}$

This is a design because each pair is in 2 blocks, $\lambda = 2$. In fact, it's an $(11, 11, 5, 5, 2)$ design.

This process only rarely works. For example, with 11 varieties, if you start with the first block $\{1, 2, 3, 4, 5\}$ then you will find that the pair $(4, 5)$ appears in 4 blocks, but the pair $(1, 6)$ appears in none. So this is not a design.

So how did we choose the first block in the previous example? This is answered by the next theorem.

THEOREM 91. *Let p be a prime of the form $4n+3$. Calculate $1^2, 2^2, \ldots, (2n+1)^2$ mod p. Then these $2n+1$ numbers (called the quadratic residues of p) will form the first block of a $(4n+3, 4n+3, 2n+1, 2n+1, n)$ design.*

We will not prove this theorem in the course, but see 'Aspects of Combinatorics' by V.W. Bryant, pp184–192.

DEFINITION 92. Designs generated by repeatedly adding 1 to all the numbers in the previous block are called *cyclic*. Ones of the form (v, v, k, k, λ) are called *symmetric*.

EXAMPLES 93. (1) Let $p = 11 = 4n+3$, where $n = 2$. Working mod 11 we have:

$$1^2 \equiv 1, \quad 2^2 \equiv 4, \quad 3^2 \equiv 9, \quad 4^2 = 16 \equiv 5, \quad 5^2 = 25 \equiv 3.$$

(With $6^2 = 36 \equiv 3$ we begin to get repeats.) So the quadratic residues of 11 are $1, 3, 4, 5, 9$ and this is how we chose the first block in Example 90.

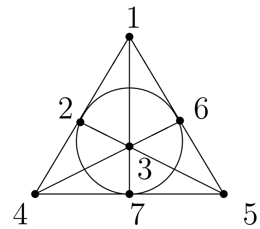
(2) Let $p = 7 = 4n+3$, with $n = 1$. Working mod 7,

$$1^2 \equiv 1, \quad 2^2 \equiv 4, \quad 3^2 \equiv 2.$$

Thus the quadratic residues of 7 are $1, 2, 4$. By Theorem 91 there is a $(7, 7, 3, 3, 1)$ design with starter block $\{1, 2, 4\}$. Here is the list of all seven blocks.

$$\{1,2,4\}$$
$$\{2,3,5\}$$
$$\{3,4,6\}$$
$$\{4,5,7\}$$
$$\{1,5,6\}$$
$$\{2,6,7\}$$
$$\{1,3,7\}$$

This design can be illustrated geometrically:



In the diagram the points correspond to the seven varieties and the lines (the six straight lines and the circle) correspond to the seven blocks. Any two points determine a line. This is a picture of something called a *finite projective plane.*

Although finite projective planes have been studied for over a century and designs only recently, it turns out that finite projective planes correspond precisely to $(n^2+n+1, n^2+n+1, n+1, n+1, 1)$ designs. (See 'Aspects of Combinatorics', by V.W. Bryant, p191.)

For which n does such a design exist? This problem is equivalent to one we have already discussed.

THEOREM 94. *There exists an* $(n^2+n+1, n^2+n+1, n+1, n+1, 1)$ *design if and only if there exist* $n-1$ *mutually orthogonal* $n \times n$ *Latin squares.*

---

2. Matrix Description of Designs

We can represent designs by matrices. This allows us to use matrix algebra to prove things about designs.

In general, given sets $A_1, A_2, \ldots, A_m \subseteq \{x_1, x_2, \ldots, x_n\}$ we can represent them by an $m \times n$ *incidence matrix* whose (i, j)th entry is

$$\begin{cases} 1, & \text{if } x_j \in A_i, \\ 0, & \text{if } x_j \notin A_i. \end{cases}$$

EXAMPLE 95. The $(9, 12, 4, 3, 1)$ coffee design of Example 87 is represented by the 12×9 matrix:

$$\begin{pmatrix} 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\ & & & & \vdots & & & & \\ & & & & \vdots & & & & \\ & & & & \vdots & & & & \end{pmatrix}$$

The columns are labelled by the coffees and the rows by the families.

In general, a (v, b, r, k, λ) design is represented by a $b \times v$ matrix, with k 1s in each row and r 1s in each column. The λ is a bit harder to spot in the matrix: any pair of columns has 1s appearing together λ times.

We now see how to test this property using matrix algebra.

THEOREM 96. *Let M be a $b \times v$ matrix of $0s$ and $1s$ with k $1s$ in each row. Then M is the matrix of a (v, b, r, k, λ) design if and only if*

$$M^T M = \begin{pmatrix} r & \lambda & \lambda & \lambda & . & . & \lambda \\ \lambda & r & \lambda & \lambda & . & . & \lambda \\ \lambda & \lambda & r & \lambda & . & . & \lambda \\ \lambda & \lambda & \lambda & r & . & . & \lambda \\ . & . & . & . & . & & . \\ . & . & . & . & & . & . \\ \lambda & \lambda & \lambda & \lambda & . & . & r \end{pmatrix}$$

Using this theorem, we can prove a result that helps establish a basic inequality.

LEMMA 97. *Let M be the matrix of a (v, b, r, k, λ) design. Then* $\det(M^T M) > 0$.

THEOREM 98 (Fisher's Inequality). *In a (v, b, r, k, λ) design, $b \geq v$.*

So to construct a design with v varieties you need at least v blocks. Thus, in a sense, the most efficient designs have $b = v$. Note that if $b = v$ then $r = \frac{bk}{v} = k$. So we have a symmetric design (v, v, k, k, λ).

EXAMPLE 99. Consider the (symmetric) $(7, 7, 3, 3, 1)$ design with blocks:

$\{1, 2, 3\}$, $\{3, 4, 5\}$, $\{1, 5, 6\}$, $\{1, 4, 7\}$, $\{2, 5, 7\}$, $\{3, 6, 7\}$, $\{2, 4, 6\}$.

Here is the matrix M of the design and its transpose.

$$M = \begin{pmatrix} 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 \end{pmatrix} \qquad M^T = \begin{pmatrix} 1 & 0 & 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 & 1 \\ 1 & 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 & 0 \end{pmatrix}$$

M^T represents the blocks $\{1, 3, 4\}$, $\{1, 5, 7\}$, $\{1, 2, 6\}$, ... and these give another $(7, 7, 3, 3, 1)$ design.

THEOREM 100. *Let M be a matrix of a symmetric design. Then M^T is also the matrix of a symmetric design.*

COROLLARY 101. *If M is the matrix of a (v, v, k, k, λ) design then any two rows of M differ in $2(k - \lambda)$ places.*

3. Error-Correcting Codes

Suppose you want to send messages in binary code.

EXAMPLE 102. Suppose you only want to send N, S, E or W. You could use the code

$$N : 00, \quad S : 01, \quad E : 10, \quad W : 11.$$

However if a single error occurs in transmission the message is changed.

One way to improve this is to add a 'check' digit.

$$N : 000, \quad S : 011, \quad E : 101, \quad W : 110.$$

The final digit was chosen so that they all have an even number of 1s. If a single error occurs the received message will not have this property. Thus it will not be an acceptable codeword and the error will be detected. The sender can be asked to repeat the message. (This principle is used in bar-codes and supermarket tills.)

Here's a bigger improvement:

$$N : 000111, \quad S : 011010, \quad E : 101100, \quad W : 110001.$$

These have been chosen so that each differs from all the others in 4 places. If up to 3 errors occur the received message will not be an acceptable codeword. So 3 errors per code word can be detected. Better still, we have a good chance of correcting an error. Suppose that at most one error occurs in transmission and the message 111001 is received. We compare it with each of the codewords:

$$\begin{array}{lllll} N: & 000111 & \rightarrow & 111001 & \text{5 changes,} \\ S: & 011010 & \rightarrow & 111001 & \text{3 changes,} \\ E: & 101100 & \rightarrow & 111001 & \text{3 changes,} \\ W: & 110001 & \rightarrow & 111001 & \text{1 change.} \end{array}$$

By using the nearest codeword we recover the correct message W.

THEOREM 103. *Let the code words of a binary code all differ in at least d places. Then*

(1) *if less than d errors per word occur in transmission, errors can be detected;*
(2) *if less than $\frac{d}{2}$ errors per word occur in transmission then errors can be corrected.*

COROLLARY 104. *Let M be the matrix of a (v, v, k, k, λ) design. Use the rows of M as the codewords of a binary code. Then*

(1) *if less than $2(k - \lambda)$ errors occur per word, errors can be detected;*
(2) *if less than $k - \lambda$ errors occur per word, errors can be corrected.*

EXAMPLE 105. Let $p = 19 = 4n + 3$, where $n = 4$. We know how to construct a cyclic $(19, 19, 9, 9, 4)$ design. For this design $k - \lambda = 9 - 4 = 5$. So, using the rows of this design's matrix as codewords gives a binary code with 19 codewords each of 19 digits, which will detect up to 9 errors per word and will correct up to 4 errors per word.

Of course, having only 19 words is very limiting, but this may be extended by clever tricks. For example, use M^* as well as M, where M^* means replace all 1s by 0s and all 0s by 1s in M. This produces a code with 38 words and good error-correcting properties.

If you find this material interesting, consider taking MAS345 Codes and Cryptography next semester!

The End

APPENDIX A

Example Sheets

This appendix contains the problems for the module, arranged into five example sheets. This is a module where it is very important to practice doing problems. I encourage you to attempt all of the problems on the sheets at some point. Some of the problems will be set for homework and marked by me or a helper. Feedback will be given, in the form of written comments on homework and also general feedback in lectures on common problems. Homework problems will be set as the module progresses. Solutions will appear on the module webpage in due course.

The following are some sources of further problems related to the module.

- Further exercises (some with solutions, some with hints, some without) on all the topics covered in the module can be found in the recommended book, "*Aspects of Combinatorics*", by Victor Bryant.
- For practice with rook polynomials, there is a Maple worksheet available from the module webpage.
- Two past exam papers with solutions are available from the module webpage.

Example Sheet 1 : The binomial coefficients

(1) Use the Binomial Theorem to show that, for any positive integer n,

$$\binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \cdots + \binom{n}{n} = 2^n$$

and

$$\binom{n}{0} - \binom{n}{1} + \binom{n}{2} + \cdots + (-1)^n \binom{n}{n} = 0.$$

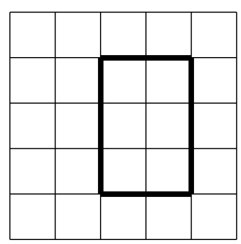
Deduce that

$$\binom{n}{0} + \binom{n}{2} + \binom{n}{4} + \cdots = \binom{n}{1} + \binom{n}{3} + \binom{n}{5} + \cdots = 2^{n-1}.$$

(2) How many different subsets (including the empty set and the whole set) are there of the set $\{1, 2, 3, \ldots, n\}$? Give brief reasons for your answer. (You might use the first part of question 1 or you might find a shorter and more direct method.)

(3) By thinking about choosing the **sides** of the rectangles, explain how many different rectangles can be seen in an $n \times n$ grid. (Rectangles of course have sides of positive length, and squares are allowed - squares are special rectangles. Position matters as well as width and height.)

The case $n = 5$ is pictured below, with one such rectangle shown in bold print.



[If you have trouble spotting a solution, you should begin by considering small examples. Work out what the answer is in the $n \times n$ case, where $n = 1, 2, 3, 4, 5$, by directly counting. Make sure you do this carefully and accurately, as mistakes here will lead to you failing to spot a pattern. Now look for a pattern and *guess* the general answer based on this. Then look for an argument that justifies your guess.]

(4) (a) If you draw n (infinite) straight lines in the plane with no two parallel and no three meeting at a point, how many intersection points will there be altogether?

(b) Now n straight lines are drawn in the plane consisting precisely of x_1 parallel in one direction, x_2 parallel in another direction, ... and x_k parallel in another direction, and with no three meeting at a point. By considering the number of intersection points 'lost' by having parallel lines, show that the number of intersection points will be

$$\frac{1}{2}(n^2 - x_1^2 - x_2^2 - \ldots - x_k^2).$$

(c) Draw 14 straight lines in the plane, with no three meeting at a point, so that there are 61 intersection points altogether.

(5) By considering colouring k items out of n items using either red or blue to colour each item, show that

$$\binom{n}{0}\binom{n}{k} + \binom{n}{1}\binom{n-1}{k-1} + \binom{n}{2}\binom{n-2}{k-2} + \cdots + \binom{n}{k}\binom{n-k}{0} = 2^k\binom{n}{k}.$$

(6) (i) Show by each of the following methods that

$$\binom{n}{0}^2 + \binom{n}{1}^2 + \binom{n}{2}^2 + \cdots + \binom{n}{n}^2 = \binom{2n}{n}.$$

(a) Consider the choice of n people from a group consisting of n men and n women.

(b) Use the expansion of $(1+x)^{2n}$.

(c) Count the number of routes in a suitable grid.

(ii) Given a collection of $2n$ people consisting of n men and n women, how many ways can a subset be chosen, the only restriction being that the number of women chosen equals the number of men chosen?

(iii) By considering the number of ways of choosing a subset of a set of n people and one person as a leader in the subset, or otherwise, show that

$$\binom{n}{1} + 2\binom{n}{2} + \cdots + n\binom{n}{n} = n2^{n-1}.$$

Example Sheet 2 : Three basic principles

(1) On an $n \times n$ board there are n^2 chess pieces, one on each square. I wish to move each piece to an adjacent square (in the same row or column) so that after all the n^2 pieces have moved there is still one piece on each square.

(a) Show that this can be done if n is even.

(b) By a colourful argument show that it cannot be done if n is odd.

(2) Any positive integer has a highest odd factor, and the number is the product of its highest odd factor with a power of 2 (e.g. 72 has highest odd factor 9, and $72 = 9 \times 2^3$). Show that, given any $n+1$ different positive integers less than or equal to $2n$, there will exist two with the same highest odd factor. Deduce that one of those numbers is a multiple of the other.

(3) Let n be a positive integer and label n pigeon-holes by $0, 1, 2, \ldots, n-1$. By placing each number of the form $1, 11, 111, \ldots$ in the pigeon-hole corresponding to its remainder when divided by n, show that there exists a number of the form $11\ldots10\ldots00$ which is a multiple of n.

(4) Prove that there exist two different powers of 7 whose difference is divisible by 1000.

(5) **Important Note:** To answer this question, you will need notation for 'the integer part of' a real number x. This is the largest integer which is less than or equal to x. I suggest using the notation $\lfloor x \rfloor$ for this. So, for example, $\lfloor 10/3 \rfloor = 3$. Some people use the notation $[x]$ and this is also fine. (Please do not write that 10/3 and 3 are equal, or any similar nonsense, as part of your answer! This comment is here because in previous years lots of people did write such things!)

Consider the set of integers $I = \{2, 3, 4, \ldots, 10000\}$. Let the primes be labelled $p_1 = 2, p_2 = 3, p_3 = 5, p_4 = 7, \ldots$ and for each $n \in I$ say that n has 'property i,' if n is divisible by p_i. (So, for example, 350 has properties 1, 3 and 4.)

(a) With the usual notation of inclusion/exclusion, show that $N(1) = 5000$ and $N(1, 2) = 1666$. State the formula for $N(1, 2, \ldots, r)$.

(b) Use the inclusion/exclusion principle to show that there are precisely 7334 numbers in I which are divisible by at least one of 2,3 and 5. (Apart from 2, 3 and 5 themselves, the other 7331 numbers are definitely not prime.)

(c) Deduce an upper bound for the number of primes less than or equal to 10000. (Clearly this bound could be decreased by including property 4, etc. This process is equivalent to the classic 'Sieve of Eratosthenes'.)

(6) To determine a function f from $\{1, 2, \ldots, m\}$ to $\{1, 2, \ldots, n\}$ you need to stipulate the values of $f(1), f(2), \ldots$ and $f(m)$. Explain briefly why there are n^m functions from $\{1, 2, \ldots, m\}$ to $\{1, 2, \ldots, n\}$.
Now consider all those functions and for $1 \leq i \leq n$ let the function f 'have property i' if none of $f(1), f(2), \ldots, f(m)$ equals i. With the usual inclusion/exclusion notation show that $N(1, 2) = (n-2)^m$. Deduce that the number of functions with **none** of the properties $1, 2, \ldots, n$ equals

$$n^m - \binom{n}{1}(n-1)^m + \binom{n}{2}(n-2)^m - \binom{n}{3}(n-3)^m + \ldots + (-1)^{n-1}\binom{n}{n-1}1^m.$$

This has counted all the functions f from $\{1, 2, \ldots, m\}$ to $\{1, 2, \ldots, n\}$ with none of the n properties; i.e. with each $i \in \{1, 2, \ldots, n\}$ equal to some $f(j)$. So what, in simple terms, have we just counted?

(7) Explain briefly why there are $\binom{n-1}{k-1}$ positive integer solutions of the equation

$$x_1 + x_2 + \cdots + x_k = n.$$

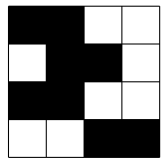
Use the Inclusion/Exclusion Principle to find the number of positive integer solutions of the equation

$$x_1 + x_2 + x_3 = 20,$$

satisfying the conditions $x_1 \leq 5$, $x_2 \leq 10$ and $x_3 \leq 15$.

Example Sheet 3 : Rook polynomials

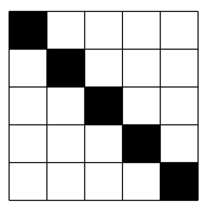
(1) Find the rook polynomial of the following (unshaded) board.



Note: If you want to practice calculating rook polynomials, there is a useful Maple worksheet on the course webpage. Simply set yourself any board and then check your answer against Maple's answer.

(2) Let n be a positive even integer and consider an $n \times n$ chess-board in which the squares are coloured black and white in the usual chequered fashion. In how many ways can n non-challenging rooks be placed on the white squares?

(3) Write down the rook polynomial of the **shaded** board illustrated below (in the general $n \times n$ case).



$\longleftarrow \quad n \quad \longrightarrow$

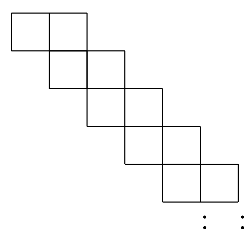
Recall that a *derangement* of $\{1, 2, \ldots, n\}$ is a permutation in which no i gets mapped to itself. Explain briefly why each derangement of $\{1, 2, \ldots, n\}$ corresponds to a layout of n non-challenging rooks on the unshaded board above. Deduce that the number of derangements of $\{1, 2, \ldots, n\}$ is

$$n! \left(\frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!} - \ldots + (-1)^n \frac{1}{n!} \right).$$

(4) Which of the following polynomials can be the rook polynomial of a board? Give reasons, including examples of appropriate boards, where possible.
a) $1 + x$,
b) $(1 + x)^n$,
c) $1 + 4x + 2x^2$,
d) $(1 + 4x + 2x^2)^2$,
e) $1 - 3x$,
f) $1 + 2x + 2x^2$.

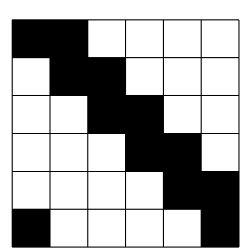
(5) Let B be the 'staircase' board illustrated below, but consisting in general of m squares. By observing that two rooks on B are in the same row or column if and only if they are adjacent, use a result from the binomial coefficients chapter to show that B has rook polynomial

$$r_B(x) = 1 + \binom{m}{1}x + \binom{m-1}{2}x^2 + \ldots + \binom{m+1-k}{k}x^k + \ldots$$



(6) This question concerns n heterosexual couples. The n men are seated in alternate seats around a circular table. The number of ways of seating the n women, with no woman next to her own partner, equals the number of ways of placing n rooks on the unshaded part of the $n \times n$ board shown below. (The special case $n = 6$ is shown.)

Use the standard technique, together with the previous question, to find the rook polynomial of the **shaded** board in the general $n \times n$ case. Deduce the number of ways of seating the n women.



$\longleftarrow \quad n \quad \longrightarrow$

Example Sheet 4 : Applications of Hall's Marriage Theorem

(1) Jobs a, b, c, d, e have to be assigned to people A, B, C, D, E, with each person getting one job. The following table gives a measure of their unsuitability for each task (the lower the number, the more suitable a person is).

	a	b	c	d	e
A	0	3	3	4	3
B	2	0	1	0	0
C	0	3	2	4	3
D	0	1	0	2	1
E	2	1	0	1	3

Use the Hungarian algorithm (with a repeated application) to assign the jobs most efficiently.

(2) Which of the following are possible scores in a tournament of 8 people?
(i) $6, 6, 5, 5, 3, 2, 1, 1$.
(ii) $6, 6, 6, 5, 2, 2, 1, 0$.
(iii) $6, 6, 5, 5, 3, 2, 1, 0$.

(3) In a tournament of n players, let the score of player i be w_i. Let l_i denote the number of games lost by player i.
(a) What is l_i (in terms of n and w_i) ?
(b) Show that $l_1, \ldots, l_n$ are the scores of a tournament.

(4) (a) What are the possible scores in a tournament of 3 players?
(b) In a tournament of n players, let w_i be the score of player i. Show that the number of sets of three players consisting of player i and two others and such that player i wins both his/her games against the other two is $\binom{w_i}{2}$.
(c) Again, in a tournament of n players, let w_i be the score of player i. Using parts (a) and (b), deduce that the number of sets of three players with the property that each player in the three won one of the three games played amongst the three is

$$\binom{n}{3} - \binom{w_1}{2} - \binom{w_2}{2} - \cdots - \binom{w_n}{2}.$$

(5) This question concerns tournaments of n players $\{P_1, P_2, \ldots, P_n\}$.

(a) How many different sets of scores are there of the games so that P_1's final score is greater than P_2's which is greater than P_3's ... which is greater than P_n's? When you have listed their scores also work out the result of each game.

(b) How many different sets of results are there of the games so that two of the players have the same score but, apart from that, all the scores are different?

(c) You are given that P_1 has the highest score and that all the other players tie second. Show that P_1 must win all his/her games and that n must be even.

Example Sheet 5 : Latin squares and designs

(1) In how many ways can the Latin rectangle

$$\begin{pmatrix} 1 & 2 & 3 & \dots & n \\ n & 1 & 2 & \dots & n-1 \end{pmatrix}$$

be extended to a $3 \times n$ Latin rectangle with entries from $\{1, 2, \dots, n\}$? (This should *not* require much new effort - use the techniques of Sheet 3.)

(2) For what value of x can the following Latin rectangle be extended to a 7×7 Latin square?
For what value of x can it be extended to a 6×6 Latin square?
Write down one such extension to a 6×6 Latin square.

$$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 1 & 2 \\ 3 & 4 & 6 & 1 \\ 4 & 1 & 2 & x \end{pmatrix}$$

(3) Let p, q and n be positive integers with $p \leq n$ and $q \leq n$. Let L be a $p \times q$ Latin rectangle in which each of the numbers $\{1, 2, \dots, n\}$ occurs the same number of times. Show that L can be extended to an $n \times n$ Latin square.

(4) Write down two orthogonal 3×3 Latin squares.

(5) Given integers v and k with $1 < k < v$ show that there exists a design with parameters $\left(v, \binom{v}{k}, \binom{v-1}{k-1}, k, \binom{v-2}{k-2}\right)$.

(6) (a) Explain briefly how to construct a $(23, 23, 11, 11, 5)$ design.
(b) Show that if a (v, b, r, k, λ) design exists (and $k \neq v-1$) then there also exists a $(v, b, b-r, v-k, b-2r+\lambda)$ design. [Hint: the idea is to replace each block by its complement in the set of varieties. You need to check that this does result in a design.]
(c) Show that, if a $(23, 23, r, k, \lambda)$ design exists, then $r = k$ and $k(k-1) = 22\lambda$. Hence find all values of r, k and λ such that a $(23, 23, r, k, \lambda)$ design exists. (Remember that to be sure that one does exist you must explain briefly how to construct it.)

(7) (a) Show that there cannot be a design with $k = 3$, $\lambda = 1$ and $v = 11$.
(b) Show that if a design has $k = 3$ and $\lambda = 1$, then v must be congruent to 1 or 3 mod 6.