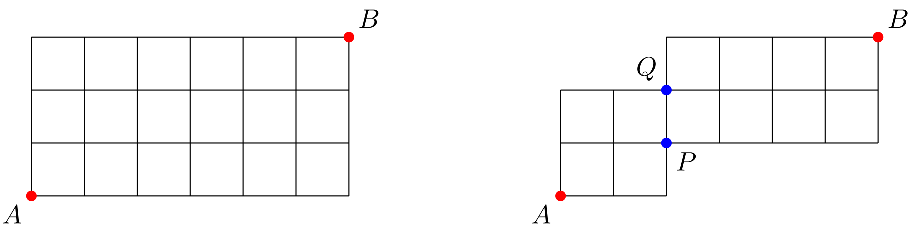
# Combinatorics Exam Solutions 2022-23

**(1)** Consider the following diagram:




We are interested in paths through this grid from $A$ to $B$ (with each path consisting of steps of length one upwards or to the right, as usual).

(a) How many paths are there from $A$ to $B$ in the left hand diagram? **(3 marks)**

(b) How many paths are there from $A$ to $B$ in the right hand diagram?
[Hint: Consider whether such paths pass through $P$ or $Q$ or both or neither.] **(7 marks)**

**Solution:**

(a) **(standard example)** A path consists of nine steps altogether, six right steps and three up steps. [1]Since the 3 up steps can be taken at any stage, there are $\binom{9}{3} = 84$ such routes. [2]

(b) **(standard type of question)** For the reason explained in (i), there are $\binom{3}{2} = 3$ paths $A$ to $P$, $\binom{6}{2} = 15$ paths $P$ to $B$, $\binom{4}{2} = 6$ paths $A$ to $Q$, just 1 from $P$ to $Q$ and $\binom{5}{1}$ from $Q$ to $B$. [3] A path $A$ to $B$ passes through $P$ or $Q$ (or both). By a simple case of Inclusion/Exclusion, the number of paths $A$ to $B$ is the number of such passing through $P$ plus the number passing through $Q$ minus the number passing through both $P$ and $Q$. [2] So this is:

$$3.15 + 6.5 - 3.1.5 = 45 + 30 - 15 = 60.\text{[2]}$$

**(2)** Find the number of integer solutions for each of the following problems:

(a) $x_1 + \cdots + x_9 = 15$ with $x_1, \ldots, x_9 \geq 0$. **(3 marks)**

(b) $x_1 \times \cdots \times x_9 = 15$ with $x_1, \ldots, x_9 \geq 0$. **(3 marks)**

(c) $x_1 + \cdots + x_9 = 5 \pmod{10}$ with $0 \leq x_1, \ldots, x_9 < 10$. **(3 marks)**

(d) $x_1^2 + \cdots + x_9^2 = 3$ with $x_1, \ldots, x_9 \in \mathbb{Z}$. **(3 marks)**

**Solution:**

(a) **(standard example)** By a standard procedure, with a horizontal grid line for each variable and measuring progress to the right along the horizontal grid lines, these solutions are in bijection with shortest paths from bottom left to top right in a 15 by $9 - 1 = 8$ grid.[2] So there are $\binom{15+8}{3} = \binom{23}{8} = 490,314$ such solutions. [1]

(b) **(Unseen. The most obvious solution is a little longer than that given here.)** A factor of 3 must appear in one of the variables $x_i$, and a factor of 5 must appear in one of the variables $x_j$ (where $j$ might be equal to $i$). There are 9 choices for $i$ and 9 choices for $j$ giving 81 solutions altogether.[3]

(c) **(similar to problem sheet example)** Here we can choose $x_1, \ldots, x_8$ arbitrarily and then set $x_9 = (5 - (x_1 + \cdots + x_8)) \pmod{10}$. Thus, the number of solutions is $10^8$. [3]

(d) **unseen.** Here three of the variables must be $\pm 1$ and the rest must be zero. There are $\binom{9}{3} = 84$ ways to choose which variables are nonzero, then $2^3 = 8$ ways to choose the $\pm$ signs, giving $2^3\binom{9}{3} = 672$ solutions altogether. [3]

**(3)** Let $k \geq 1$ and $n \geq 3k-1$. This question concerns seating $k$ couples in a row of $n$ seats. The couples are $c_i = (c_i^L, c_i^R)$ for $1 \leq i \leq k$ and we want to seat them according to the following rules.

- For each $i$, $c_i^L$ sits in the adjacent seat to the left of $c_i^R$.
- The couples are in order $c_1, c_2, \ldots, c_k$ from left to right.
- Different couples are not adjacent: there is a gap of at least one seat between one couple and another.

Let $T_n^k$ denote the number of ways this can be done.

(a) What is $T_n^k$? Give a direct argument for your answer. **(4 marks)**

(b) Show, from the description of the seating problem, that $T_{3k-1}^k = 1$ and that $T_n^k = T_{n-1}^k + T_{n-3}^{k-1}$ for $n \geq 3k$. **(5 marks)**

(c) Check that your answer to part (i) is consistent with (ii). **(3 marks)**

**Solution:**

(a) **(unseen, related gappy problems seen)** We need to choose $k$ positions for the couples (in the specified order and with no choice about which way round a couple sits) from the spaces, including the ends, between the remaining $n-2k+1$ places. So $T_n^k = \binom{n-2k+1}{k}$. [4]

**[Students may well give a bijective argument to a similar gappy problem instead; that's fine if correct.]**

(b) **(unseen)** If we have $3k-1$ seats, there is just one way to place the couples, with the first couple at the left hand end, exactly one seat between couples and the last pair at the right hand end. So $T_{3k-1}^k = 1$. [1]

For the recurrence, either the $n$th seat in the row is occupied or it is not. [1]If it is not, we have to place the $k$ couples (according to the rules) in the remaining $n-1$ seats, which can be done in $T_{n-1}^k$ ways. [1]If it is occupied, so is seat $n-1$ and seat $n-2$ must be unoccupied. So we have to place the remaining $k-1$ couples (according to the rules) in the remaining $n-3$ seats, which can be done in $T_{n-3}^{k-1}$ ways. [1]So $T_n^k = T_{n-1}^k + T_{n-3}^{k-1}$ for $n \geq 3k$. [1]

(c) **(unseen)** We have $\binom{(3k-1)-2k+1}{k} = \binom{k}{k} = 1$. [1]And

$$\begin{aligned}\binom{(n-1)-2k+1}{k} + \binom{(n-3)-2(k-1)+1}{k-1} &= \binom{n-2k}{k} + \binom{n-2k}{k-1} \\ &= \binom{n-2k+1}{k}\end{aligned}$$

where the last step is Pascal's identity.

Thus $T_n^k = \binom{n-2k+1}{k}$ satisfies the recurrence in (ii). [2]

**(4)**

(a) State the Pigeonhole Principle. **(2 marks)**

(b) Let $X$ be a set of 11 numbers from $\{1, 2, \ldots, 80\}$. Show that there exist two different subsets of $X$ each having exactly 4 elements and such that the sum of their elements is the same. **(5 marks)**

**Solution:**

(a) **(bookwork)** If more than $n$ items are placed in $n$ pigeon-holes, then some pigeon-hole will contain more than one item. [2]

**[Other formulations are fine.]**

(b) **(standard type of question)** The number of 4 element subsets of a set with 11 elements is $\binom{11}{4} = 330$. [1]The largest possible sum of elements of such a subset is $80 + 79 + 78 + 77 = 314$. [1]So all the sums are in the range $0, 1, \ldots, 314$, giving at most 315 possibilities. [1]Label pigeonholes by possible sums and assign each subset to the pigeonhole labelled by its sum. Since there are more subsets than pigeonholes, the PHP says that there are two in the same pigeonhole, that is, with the same sum. [2]

**(5)**

(a) State the positive form of the Inclusion/Exclusion Principle. **(3 marks)**

(b) Use the Inclusion/Exclusion Principle to find the number of permutations of the numbers $1, 2, \ldots, 10$ such that at least one even number is fixed. **(7 marks)**

**Solution:**

(a) **(bookwork)** Let $B_a$ be a subset of a set $B$, for $a \in A$. Write $B_I = \bigcap_{i \in I} B_i$ for $I \subseteq A$. Then $|\cup_{a \in A} B_a| = \sum_{I \neq \emptyset} (-1)^{|I|+1} |B_I|$. [3]

[ **Alternative correct formulations are fine, including specialising to** $A = \{1, 2, \ldots, n\}$**.** ]

(b) **(standard type of question)** For $1 \leq i \leq 5$, let $P_i$ denote the subset of permutations of $\{1, 2, \ldots, 10\}$ fixing $2i$. We adopt standard I/EP notation, so for example $P_{i,j}$ denotes the subset of permutations of $\{1, 2, \ldots, 10\}$ fixing $2i$ and $2j$. [2] Then $|P_i| = 9!$ (since we have a permutation of the other 9 numbers) for each $i$. Similarly, $|P_{i,j}| = 8!$ for all $i, j$, $|P_{i,j,k}| = 7!$ for all $i, j, k$, $|P_{i,j,k,l}| = 6!$ for all $i, j, k, l$ and $|P_{i,j,k,l,m}| = 5!$ for all $i, j, k, l, m$. [3]

We want the number of elements in the union of all these subsets and the I/EP gives

$$\binom{5}{1} 9! - \binom{5}{2} 8! + \binom{5}{3} 7! - \binom{5}{4} 6! + \binom{5}{5} 5!$$
$$= 5.9! - 10.8! + 10.7! - 5.6! + 5! = 1,458,120 [2]$$

**(6)**

(a) Let $n \geq 3$. Find the rook polynomial of the full $n \times 3$ board. **(4 marks)**

(b) Which of the following polynomials can be the rook polynomial of a board? Give reasons for your answers, including examples of appropriate boards.

(i) $1 - 7x$.

(ii) $(1 + x)(1 + 4x + 2x^2)^2$.

(iii) $1 + \binom{n}{1} x + \binom{n}{2} x^2 + \cdots + \binom{n}{n} x^n$. **(5 marks)**

**Solution:**

(a) **(This is a special case of a proposition in the notes, but it's easy to do by hand.)**

The rook polynomial is of the form $1 + a_1 x + a_2 x^2 + a_3 x^3$, where $a_i$ is the number of ways of placing $i$ non-challenging rooks and this is zero if $i > 3$ as we only have 3 rows. [1]

We have $a_1 = 3n$, since this is the number of available squares and one rook can be placed on any square. [1]

We have $a_2 = 3n(n - 1)$, since there 3 ways to pick two rows from three for two rooks, and then the first rook can be in any of the $n$ squares in its row and the second in any of the $n - 1$ squares in its row not challenging the first rook. [1]

And $a_3 = n(n - 1)(n - 2)$ since we have $n$ choices to place a rook on the first row, leaving $n - 1$ for the second and $n - 2$ for the third. So the rook polynomial is $1 + 3nx + 3n(n - 1)x^2 + n(n - 1)(n - 2)$. [1]

(b) **(unseen)**

(i) No, the coefficient of $x$ is the number of squares and so can't be negative. [1]

(ii) Yes, for example a board comprising two disjoint full $2 \times 2$ boards and a disjoint $1 \times 1$ board. [2]

(iii) Yes, this is $(1 + x)^n$ and so is the rook polynomial of any board comprising $n$ disjoint $1 \times 1$ boards. [2]

**(7)** Recall that a derangement of $\{1, 2, \ldots, n\}$ is a permutation leaving none of the numbers fixed. We write $d_n$ for the number of derangements of $\{1, 2, \ldots, n\}$.

(a) Show that

$$\sum_{k=0}^{n} \binom{n}{k} d_{n-k} = n!.$$

**(4 marks)**

(b) Show that, for $n \geq 3$,

$$d_n = (n-1)(d_{n-2} + d_{n-1}).$$

**(6 marks)**

**Solution:**

(a) **(unseen)** The number of permutations of $\{1, 2, \ldots, n\}$ is $n!$ and each permutation fixes some number of integers $k$, with $0 \leq k \leq n$ [1]The number of permutations fixing a given $k$ integers (and none of the rest) is equal to the number of derangements of the rest, $d_{n-k}$ [1]. So the number of permutations fixing exactly $k$ integers is equal to the number of ways of choosing the $k$ integers times the number of derangements of the rest, $\binom{n}{k} d_{n-k}$ [1]. Thus $\sum_{k=0}^{n} \binom{n}{k} d_{n-k} = n!$. [1]

(b) **(unseen)** There are $n-1$ possibilities for the image of 1 under a derangement, and the same number of derangements sending 1 to each of these $n-1$ possibilities [1]. Therefore $d_n = (n-1)d'_n$, where $d'_n$ is the number of derangements sending 1 to 2 [1]. A derangement sending 1 to 2 either sends 2 to 1 or 2 to some other number [1]. In the first case, the number of such is equal to the number of derangements of $\{3, 4, \ldots, n\}$, that is, $d_{n-2}$ [1]. In the second case, the number of such is equal to the number of derangements of $\{1, 3, 4, \ldots, n\}$, that is, $d_{n-1}$ [1]. So $d'_n = d_{n-2} + d_{n-1}$ and $d_n = (n-1)(d_{n-2} + d_{n-1})$ [1].

**(8)** Suppose that we have two tournaments, each of $2n$ players, where the scores are $T_i$ and $U_i$, for $1 \leq i \leq 2n$. Show that there is a tournament of $4n$ players with scores $T_i + n$, $U_i + n$, for $1 \leq i \leq 2n$. **(5 marks)**

**Solution: (unseen)**

We extend the two given tournaments to a tournament of all the $4n$ players. To do so, we need to specify the outcome of each new game where the player with score $T_i$ (say $t_i$) plays the player with score $U_j$ (say $u_j$) [2].

We can do this by saying that $t_i$ wins if $i = j \pmod 2$ and $u_j$ wins if $i \neq j \pmod 2$. This means that each player wins half of their $2n$ new games and so each new score is the old score plus $n$ as required [3].

[**Any correct argument gets the marks. Students may use Landau's theorem.** ]

**(9)**

(a) Explain what it means for two $n \times n$ Latin squares with $P = Q = N = \{1, 2, \ldots, n\}$ to be orthogonal. **(2 marks)**

(b) Prove that there exist at most $n-1$ mutually orthogonal $n \times n$ Latin squares. **(8 marks)**

**Solution:**

(a) **(bookwork)** Two $n \times n$ Latin squares with $P = Q = N = \{1, 2, \ldots, n\}$, say $L$ and $M$, are orthogonal if the pairs $(L_{ij}, M_{ij})$ run through all possible $n^2$ pairs [2].

(b) **(bookwork)** Let $L_1, L_2, \ldots, L_q$ be mutually orthogonal $n \times n$ Latin squares. We need to show that $q \leq n-1$ [1]. If the first row of $L_1$ is $(a_1, a_2, \ldots, a_n)$, then replace $a_i$ by $i$ throughout $L_1$ to give $L'_1$ [1]. Now $L'_1$ is still a Latin square and it is straightforward to see that it is still orthogonal to all the rest [1]. Repeat this process for $L_2, \ldots, L_q$ [1]. Then we have $q$ mutually orthogonal Latin squares $L'_1, L'_2, \ldots, L'_q$, all with first row $(1, 2, \ldots, n)$ [1]. In each of these $q$ Latin squares, consider the $(2,1)$ entry. Because the squares are Latin none of these is a 1, so these entries are all in $\{2, \ldots, n\}$ [1]. Also, they are all different, because a repeat of $x$, say, in this position in $L'_i$ and $L'_j$ would mean that the pair $(x, x)$ occurs twice among the $\big((L'_i)_{ab}, (L'_j)_{ab}\big)$, corresponding to position $(1, x)$ and to position $(2, 1)$, contradicting the orthogonality of $L'_i$ and $L'_j$ [1]. So the $q$ entries in the $(2,1)$ positions of the squares are different elements of $\{2, \ldots, n\}$ and so $q \leq n-1$ as required [1].

**(10)**

(a) In a $(v, b, r, k, \lambda)$-block design, the number of varieties is $v$ and the number of blocks is $b$. Explain the meaning of each of the other parameters. **(3 marks)**

(b) State two equations relating $r$ to the other parameters of a design. **(2 marks)**

(c) Consider all choices of 3 numbers from $\{1, 2, \ldots, 6\}$. Show that these form the blocks of a design and determine the parameters. **(5 marks)**

(d) Let $2 \le i \le n$. Show that there is a design with parameters

$$\left(n, \binom{n}{i}, \binom{n-1}{i-1}, i, \binom{n-2}{i-2}\right).$$

**(5 marks)**

**Solution:**

(a) **(bookwork)** The number of blocks of the design each variety appears in is $r$ [1]. The number of varieties per block is $k$ [1]. The number of blocks of the design each pair of varieties appears in is $\lambda$ [1].

(b) **(bookwork)**

$$r = \frac{bk}{v} = \frac{\lambda(v-1)}{k-1} \text{[2]}.$$

(c) **(unseen)** There will be $\binom{6}{3} = 20$ blocks corresponding to the ways of choosing 3 numbers from $\{1, 2, \ldots, 6\}$ [1]. Clearly, we have $v = 6$ and $k = 3$ [1]. Each number appears in $\binom{5}{2} = 10$ blocks since that is the number of ways to choose 2 other numbers to complete the block [1].

A given pair appears in $\binom{4}{1} = 4$ blocks, since one can choose any of the remaining 4 numbers to complete the block. Since this is the same for all pairs, we have a design with $\lambda = 4$ [2]. So the parameters are $(6, 20, 10, 3, 4)$.

(d) **(unseen)** Consider all choices of $i$ numbers from $\{1, 2, \ldots, n\}$. These form the blocks [2]. As in the case of $n = 6, i = 3$ in (c), we have $n$ numbers (varieties), $\binom{n}{i}$ blocks, each variety in $\binom{n-1}{i-1}$ blocks, $i$ varieties per block and each pair in $\binom{n-2}{i-2}$ blocks. Therefore we have a design, with the claimed parameters [3].

[ **This is the obvious thing to do, generalising (c), but any correct justification gets the marks.** ]