University of Sheffield

Ancillary Material:

None.

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School of Mathematics and Statistics
Autumn Semester 2023–24

Module Code and Title:
MAS334 Combinatorics

Exam Duration:

2 hours 30 minutes

Exam Instructions:

Attempt all the questions. Give justification for all numerical answers. The allocation of marks is shown in brackets.

Registration number from U-Card (9 digits) – to be completed by candidate

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1 (a) State Pascal's Identity for binomial coefficients. ***(2 marks)***

(b) Suppose that r, s, k are integers with $0 \le k \le r \le s$. Using proof by induction on s, or otherwise, show that

$$\sum_{m=r}^{s} \binom{m}{k} = \binom{s+1}{k+1} - \binom{r}{k+1}.$$

(5 marks)

2 Consider the equation

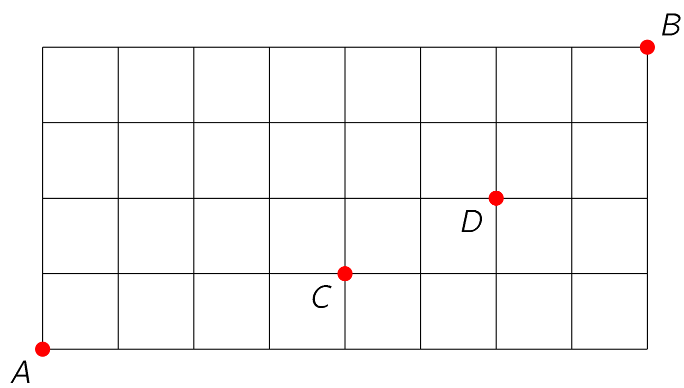
$$x_1 + x_2 + \cdots + x_k = n.$$

(a) How many solutions are there of this equation in which each x_i is a non-negative integer? Give a brief reason for your answer. ***(3 marks)***

(b) How many of the solutions have each x_i equal to 0 or 1? ***(3 marks)***

(c) For a positive integer r, how many of the solutions have x_r as the first positive number in the list $x_1, x_2, \ldots, x_k$? ***(3 marks)***

3 This question concerns routes in the grid illustrated:



(a) How many routes are there from A to B along the lines of the grid (always moving up or to the right, as usual)? Give a brief reason for your answer. ***(3 marks)***

(b) Find the number of such routes which do not pass through C or D. ***(8 marks)***

4 (a) Suppose that you are given 22 (not necessarily different) integers such that when you multiply them together you get 1. Show that when you add them up it is impossible to get 0. ***(4 marks)***

(b) Suppose that the numbers 1 to 10 are written in a row, and between each adjacent pair of numbers we insert either a plus sign or a minus sign, giving an expression such as $1-2-3+4+5-6+7-8+9-10$. Is it possible to choose the plus and minus signs in such a way that the value of the resulting expression is zero? ***(4 marks)***

5 (a) State the Pigeonhole Principle. ***(2 marks)***

(b) Show that there exists an integer whose decimal representation consists entirely of 1s (that is, an integer of the form $11\cdots11$) which is divisible by $13 \times 17 \times 19$.
[**Hint:** as well as numbers of the form $11\cdots11$, it may be helpful to consider numbers of the form $11\cdots10\cdots00$.] ***(5 marks)***

6 (a) State the Inclusion/Exclusion Principle. ***(3 marks)***

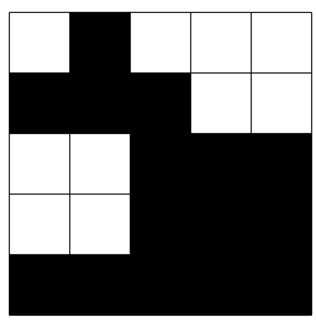
(b) Put $N = \{1, 2, 3, 4, 5, 6, 7, 8, 9\}$ and $M = \{3, 6, 9\}$. Use the Inclusion/Exclusion Principle to find the number of permutations of the set N that fix at least one member of M. ***(6 marks)***

7 (a) Let B be part of an $n \times n$ board, and let C and D be subsets of B.

(i) Explain what it means to say that B is the fully disjoint union of C and D. ***(3 marks)***

(ii) If B is the fully disjoint union of C and D, what is the relationship between the corresponding rook polynomials? ***(1 mark)***

(b) Calculate the rook polynomial of the (unshaded) board B:



(8 marks)

(c) For each of the following polynomials $p_k(x)$, either find a board B_k whose rook polynomial is $p_k(x)$, or explain why that is not possible. ***(8 marks)***

$$p_1(x) = 1 + 10x - 3x^2 + x^3 \qquad p_2(x) = 4 + 3x + 2x^2 + x^3$$
$$p_3(x) = 1 + 4x + 4x^3 + x^4 \qquad p_4(x) = 1 + 16x + 72x^2 + 96x^3 + 24x^4$$
$$p_5(x) = 1 + 8x + 14x^2 + 4x^3 \qquad p_6(x) = 1 + 4x + 6x^2 + 4x^3 + x^4$$

8 (a) State Landau's theorem on scores in tournaments. ***(4 marks)***

(b) Suppose we have a tournament with 6 players, in which 3 players score x and the other 3 players score y, where $x > y$. What are the possible values of x and y? ***(4 marks)***

(c) For each of the pairs (x, y) that you found in (b), give an example of a corresponding tournament. ***(4 marks)***

9 Let p, n be integers with $0 < p < n$. Let L be a $p \times (n-p)$ latin rectangle with entries in $\{1, \ldots, n\}$. Using an appropriate theorem from the notes, show that X can be extended to an $n \times n$ latin square. ***(5 marks)***

10 (a) Let $p = 4m + 3$ be a prime number. Explain how to use quadratic residues modulo p to construct a block design with parameters $(4m+3, 4m+3, 2m+1, 2m+1, m)$. You should explain the key facts that need to be proved to verify that your construction works, but you do not need to prove any of them. ***(7 marks)***

(b) Consider a (v, b, r, k, λ) design. Give two equations expressing r in terms of the other parameters of the design. ***(2 marks)***

(c) Do there exist designs with the following parameters? Give brief reasons for your answers.

(i) $(11, 11, 5, 5, 2)$.

(ii) $(11, 11, 4, 6, 2)$.

(iii) $(11, 11, 6, 6, 2)$.

(3 marks)

End of Question Paper