Combinatorics Exam Solutions 2023-24

(1)

(a) State Pascal's Identity for binomial coefficients. **(2 marks)**

(b) Suppose that r, s, k are integers with $0 \leq k \leq r \leq s$. Using proof by induction on s, or otherwise, show that

$$\sum_{m=r}^{s} \binom{m}{k} = \binom{s+1}{k+1} - \binom{r}{k+1}.$$

(5 marks)

Solution:

(a) Pascal's relation says that for $n, k \geq 0$ with $(n, k) \neq (0, 0)$ we have $\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}$. [2] **Bookwork**

(b) We argue by induction on s starting with $s = r$. If $s = r$ then the claim is that $\binom{s}{k} = \binom{s+1}{k} - \binom{s}{k+1}$, which is a rearrangement of Pascal's identity. For $s > r$, we can assume inductively that

$$\sum_{m=r}^{s-1} \binom{m}{k} = \binom{s}{k+1} - \binom{r}{k+1}.$$

By adding $\binom{s}{k}$ to both sides and using Pascal's identity again, we get

$$\sum_{m=r}^{s} \binom{m}{k} = \binom{s}{k} + \binom{s}{k+1} - \binom{r}{k+1} = \binom{s+1}{k+1} - \binom{r}{k+1},$$

as required. [5] **Unseen**

As a different approach, let A_m be the set of subsets $U \subseteq \{1, \ldots, s+1\}$ with $|U| = k+1$ and $\max(U) = m+1$. Put $B = A_r \cup \cdots \cup A_s$. Then $|A_m| = \binom{m}{k}$ so the left hand side of our equation is $|B|$. On the other hand, the right hand side is the number of subsets $U \subseteq \{1, \ldots, s+1\}$ that are not contained in $\{1, \ldots, r\}$, which is $|B|$ again.

(2) Consider the equation

$$x_1 + x_2 + \cdots + x_k = n.$$

(a) How many solutions are there of this equation in which each x_i is a non-negative integer? Give a brief reason for your answer. **(3 marks)**

(b) How many of the solutions have each x_i equal to 0 or 1? **(3 marks)**

(c) For a positive integer r, how many of the solutions have x_r as the first positive number in the list $x_1, x_2, \ldots, x_k$? **(3 marks)**

Solution:

(a) The number of solutions is $\binom{n+k-1}{k-1}$ [1]. Indeed, this is the number of binary sequences of length $n+k-1$ containing $k-1$ ones. These ones separate the remaining n zeros into k blocks of lengths $x_1, \ldots, x_k \geq 0$ with $\sum_i x_i = n$. This gives a bijection between these binary sequences and solutions of the equation. [2] **Bookwork**

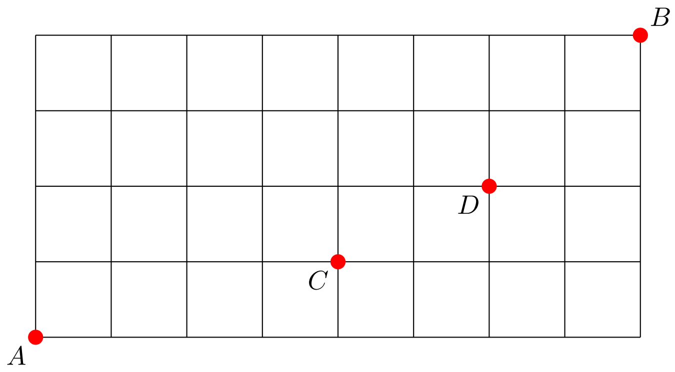
(b) Now consider solutions in which each x_k lies in $\{0,1\}$. We just need to choose n of the k variables to be equal to one, and the rest wil be zero. It follows that there are $\binom{k}{n}$ solutions. [3] **Similar examples seen**

(c) Now fix r and consider solutions x where $x_r > 0$ but $x_i = 0$ for $i < r$. If we put $x'_r = x_r - 1 \geq 0$, these are equivalent to solutions of $x'_r + x_{r+1} + \cdots + x_k = n-1$ with all variables nonnegative. Here the number of variables is $k' = k - r + 1$ so the number of solutions is

$$\binom{n-1+k'-1}{k'-1} = \binom{n+k-r-1}{k-r}.$$ [3]

Similar examples seen

(3) This question concerns routes in the grid illustrated:



(a) How many routes are there from A to B along the lines of the grid (always moving up or to the right, as usual)? Give a brief reason for your answer. **(3 marks)**

(b) Find the number of such routes which do not pass through C or D. **(8 marks)**

Solution:

(a) The number of possible routes from A to B is $n(A,B) = \binom{12}{8} = 495$. **[1]**Indeed, we need to take 8 horizontal steps and 4 vertical steps, and the route is specified by choosing which 8 of the 12 steps are to be horizontal. **[2]**
Bookwork

(b) **Answers in terms of unevaluated binomial coefficients will be accepted. Similar examples have been seen.**

- The number of routes passing through C is $n(A,C)n(C,B) = \binom{5}{4}\binom{7}{4} = 175$. **[2]**
- The number of routes passing through D is $n(A,D)n(D,B) = \binom{8}{6}\binom{4}{2} = 168$. **[2]**
- The number of routes passing through both C and D is $n(A,C)n(C,D)n(D,B) = \binom{5}{4}\binom{3}{2}\binom{4}{2} = 90$. **[2]**
- Thus, by the IEP, the number of paths not passing through C or D is $495 - 175 - 168 + 90 = 242$. **[2]**

(4)

(a) Suppose that you are given 22 (not necessarily different) integers such that when you multiply them together you get 1. Show that when you add them up it is impossible to get 0. **(4 marks)**

(b) Suppose that the numbers 1 to 10 are written in a row, and between each adjacent pair of numbers we insert either a plus sign or a minus sign, giving an expression such as $1-2-3+4+5-6+7-8+9-10$. Is it possible to choose the plus and minus signs in such a way that the value of the resulting expression is zero? **(4 marks)**

Solution:

(a) Suppose we have $x_1, \dots, x_{22} \in \mathbb{Z}$ with $\prod_i x_i = 1$. This is only possible if each of the numbers x_i is ± 1 **[1]**. Suppose that m of the numbers are -1, and the other $22-m$ are $+1$ **[1]**. Because $\prod_i x_i = 1$, we see that m must be even, say $m = 2k$ **[1]**. Now $\sum_i x_i = 2k \times (-1) + (22-2k) \times (+1) = 22 - 4k$. As 22 is not divisible by 4, this cannot be zero **[1]**. **Unseen**

(b) If we had an equation of the form $\pm 1 \pm 2 \pm \cdots \pm 10 = 0$, we could reduce it mod 2 (remembering that $-k = k \pmod 2$) to get $1 + 2 + \cdots + 10 = 0 \pmod 2$ **[2]**. However, by the standard arithmetic progression formula the left hand side is $\binom{11}{2} = 55$, which is not zero mod 2 **[2]**. **Unseen**

(5)

(a) State the Pigeonhole Principle. **(2 marks)**

(b) Show that there exists an integer whose decimal representation consists entirely of 1s (that is, an integer of the form $11\cdots 11$) which is divisible by $13 \times 17 \times 19$.
[**Hint:** as well as numbers of the form $11\cdots 11$, it may be helpful to consider numbers of the form $11\cdots 10\cdots 00$.]
(5 marks)

Solution:

(a) Suppose we have a set B with $|B| = n$, and subsets $A_1, \ldots, A_m \subseteq B$ with $B = A_1 \cup \cdots \cup A_m$. Suppose also that $m < n$; then there exists i such that $|A_i| > 1$. [2]
Bookwork. Full marks will be given for any correct statement of the same general type.

(b) Put $d = 13 \times 17 \times 19$ and $D = \{0, 1, \ldots, d-1\}$ so $|D| = d$. Let a_n be the number $11\cdots 11$ with n ones and put $b_n = a_n \pmod{d} \in D$. The numbers $b_1, \ldots, b_{d+1}$ cannot all be distinct, so we can choose $n < m$ with $b_n = b_m$, which means that $a_m - a_n$ is divisible by d [2]. Now $a_m - a_n$ consists of $m - n$ ones followed by n zeros, i.e. $a_m - a_n = 10^n a_{m-n}$, so d divides $10^n a_{m-n}$ [1]. It is clear that the numbers $d = 13 \times 17 \times 19$ and $10^n = 2^n \times 5^n$ are coprime, so d must divide a_{m-n}, as required [2].

Alternatively, one can use the formula $a_n = (10^n - 1)/9$ together with Fermat's Little Theorem to prove that a_n is divisible by d whenever n id divisible by the least common multiple of $13 - 1$, $17 - 1$ and $19 - 1$, which is 144.

This is fairly similar to other pigeonhole arguments about congruence that have been seen.

(6)

(a) State the Inclusion/Exclusion Principle. **(3 marks)**

(b) Put $N = \{1, 2, 3, 4, 5, 6, 7, 8, 9\}$ and $M = \{3, 6, 9\}$. Use the Inclusion/Exclusion Principle to find the number of permutations of the set N that fix at least one member of M. **(6 marks)**

Solution:

(a) Consider a finite set B, with a list of subsets $B_1, \dots, B_n \subseteq B$. For $I \subseteq \{1, \dots, n\}$ put $B_I = \bigcap_{i \in I} B_i$, with the convention that $B_\emptyset = B$. The IEP says that

$$|B_1 \cup \cdots \cup B_n| = \sum_{I \neq \emptyset} (-1)^{|I|-1} |B_I|, \text{[3]}$$

or equivalently

$$|B \setminus (B_1 \cup \cdots \cup B_n)| = \sum_{I} (-1)^{|I|} |B_I|.$$

Bookwork. Full marks will be given for either of the equivalent forms. Versions with ellipses instead of summation notation will be accepted if they are sufficiently clear.

(b) Let P be the set of all permutations of N, so $|P| = 9!$ **[1]**. Let P_k be the subset of permutations that send k to itself **[1]**, and put $Q = P_3 \cup P_6 \cup P_9$. We need to find $|Q|$. By the IEP, this is

$$|Q| = |P_3| + |P_6| + |P_9| - |P_{36}| - |P_{39}| - |P_{69}| + |P_{369}|.\text{[1]}$$

The set P_3 is essentially the set of permutations of $N \setminus \{3\}$, so $|P_3| = 8!$. Similarly, $|P_6| = |P_9| = 8!$ **[1]**. The set P_{36} is essentially the set of permutations of $N \setminus \{3, 6\}$, so $|P_{36}| = 7!$. Similarly, $|P_{39}| = |P_{69}| = 7!$ **[1]**. The set P_{36} is essentially the set of permutations of $N \setminus \{3, 6, 9\}$, so $|P_{369}| = 6!$. We therefore have

$$\begin{aligned} |Q| &= 3 \times 8! - 3 \times 7! + 6! = (3 \times 8 \times 7 - 3 \times 7 + 1) \times 6! = 148 \times 720 \\ &= 106560\text{[1]} \end{aligned}$$

Standard application of IEP

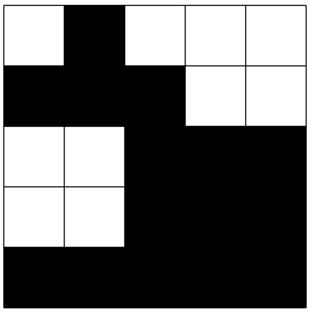
(7)

(a) Let B be part of an $n \times n$ board, and let C and D be subsets of B.

  (i) Explain what it means to say that B is the fully disjoint union of C and D. **(3 marks)**

  (ii) If B is the fully disjoint union of C and D, what is the relationship between the corresponding rook polynomials? **(1 marks)**

(b) Calculate the rook polynomial of the (unshaded) board B:



(8 marks)

(c) For each of the following polynomials $p_k(x)$, either find a board B_k whose rook polynomial is $p_k(x)$, or explain why that is not possible. **(8 marks)**

$$p_1(x) = 1 + 10x - 3x^2 + x^3 \qquad p_2(x) = 4 + 3x + 2x^2 + x^3$$
$$p_3(x) = 1 + 4x + 4x^3 + x^4 \qquad p_4(x) = 1 + 16x + 72x^2 + 96x^3 + 24x^4$$
$$p_5(x) = 1 + 8x + 14x^2 + 4x^3 \qquad p_6(x) = 1 + 4x + 6x^2 + 4x^3 + x^4$$

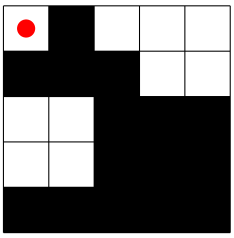
Solution:

(a) (i) **Bookwork.** We say that B is the fully disjoint union of C and D if

- $B = C \cup D$
- No row meets both C and D
- No column meets both C and D. [3]

  (ii) If so, then $r_B(x) = r_C(x) r_D(x)$. [1]

(b) We call the original board B_1 and introduce additional boards as follows:

1


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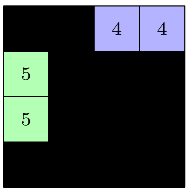

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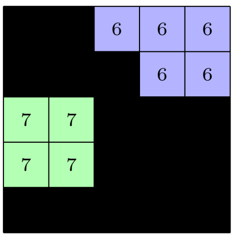

Let $r_k(x)$ be the rook polynomial of B_k. We have

$$\begin{aligned}
r_4(x) &= r_5(x) = 1 + 2x[1] \\
r_6(x) &= 1 + 5x + 4x^2[1] \\
r_7(x) &= 1 + 4x + 2x^2[1] \\
r_3(x) &= r_6(x)\, r_7(x) = 1 + 9x + 26x^2 + 26x^3 + 8x^4[1] \\
r_2(x) &= r_4(x)\, r_5(x) = 1 + 4x + 4x^2[1] \\
r_1(x) &= r_3(x) + x\, r_2(x) = 1 + 10x + 30x^2 + 30x^3 + 8x^4.[1]
\end{aligned}$$

(Indeed, the polynomials $r_4(x)$ to $r_7(x)$ are obtained by inspection, then $r_3(x) = r_6(x)r_7(x)$ because B_3 is the fully disjoint union of B_6 and B_7, and $r_2(x) = r_4(x)r_5(x)$ because B_2 is the fully disjoint union of B_4 and B_5, and $r_1(x) = r_3(x)+x\,r_2(x)$ by the blocking and stripping theorem. [2]) Thus, the rook polynomial of the original board B is $1 + 10x + 30x^2 + 30x^3 + 8x^4$. **Standard calculation.**

(c)
- $p_1(x)$ cannot be a rook polynomial because it has a negative coefficient. [1]
- $p_2(x)$ cannot be a rook polynomial because the constant term is not equal to one. [1]
- $p_3(x)$ also cannot be the rook polynomial of any board B. Indeed, the coefficient of x^2 is zero, which would mean that it it is not possible to place two non-challenging rooks, which would also mean that it is not possible to place three non-challenging rooks, contradicting the fact that the coefficent of x^3 is nonzero. [2]
- $p_4(x)$ is $\sum_{k=0}^{4} \binom{4}{k}^2 k! x^k$, which is the rook polynomial of a full 4×4 board. [1]
- $p_5(x)$ is the rook polynomial of a 3×3 board with a single square blocked off. [2]
- $p_6(x)$ is $(1 + x)^4$, which is the rook polynomial of a 4×4 board in which only the diagonal squares are unblocked. [1]

Similar examples have been seen, but not for the case of $p_3(x)$.

(8)

(a) State Landau's theorem on scores in tournaments. **(4 marks)**

(b) Suppose we have a tournament with 6 players, in which 3 players score x and the other 3 players score y, where $x > y$. What are the possible values of x and y? **(4 marks)**

(c) For each of the pairs (x, y) that you found in (b), give an example of a corresponding tournament. **(4 marks)**

Solution:

(a) **Bookwork.** Landau's theorem is as follows. Consider a list $s_1, \dots, s_n$ of nonnegative integers with $\sum_i s_i = \binom{n}{2}$ [1]. Then the following conditions are equivalent:

(1) There is an n-player tournament in which player i wins s_i games for all i [1]

(2) The sum of any k of the terms s_i is at least $\binom{k}{2}$ [1]

(3) The sum of any k of the terms s_i is at most $\binom{k}{2} + k(n-k)$. [1]

(b) **Similar problems have been seen.** Suppose that the score sequence (in decreasing order) is (x, x, x, y, y, y). By Landau's theorem, we must have

$$y \geq \binom{1}{2} = 0 \qquad 2y \geq \binom{2}{2} = 1 \qquad 3y \geq \binom{3}{2} = 3$$

$$x + 3y \geq \binom{4}{2} = 6 \qquad 2x + 3y \geq \binom{5}{2} = 10 \qquad 3x + 3y = \binom{6}{2} = 15.\text{[2]}$$

The final equation $3x + 3y = 15$ gives $y = 5 - x$. As $x > y$ we have $x > 5 - x$ so $x \geq 3$. The third relation $3y \geq 3$ gives $y \geq 1$ so $x \leq 4$. We thus have $(x, y) = (3, 2)$ or $(x, y) = (4, 1)$ [2]. It is easy to see that all six of the above relations hold in these cases.

(c) **Similar problems have been seen.** One possible solution is as follows: [4]

	0	1	2	3	4	5
0		W	L	W	W	W
1	L		W	W	W	W
2	W	L		W	W	W
3	L	L	L		W	L
4	L	L	L	L		W
5	L	L	L	W	L	

scores $(4, 4, 4, 1, 1, 1)$

	0	1	2	3	4	5
0		W	W	W	L	L
1	L		W	W	W	L
2	L	L		W	W	W
3	L	L	L		W	W
4	W	L	L	L		W
5	W	W	L	L	L	

scores $(3, 3, 3, 2, 2, 2)$

Full credit will be given for correct examples constructed by any means.

(9) Let p, n be integers with $0 < p < n$. Let L be a $p \times (n-p)$ latin rectangle with entries in $\{1, \ldots, n\}$. Using an appropriate theorem from the notes, show that X can be extended to an $n \times n$ latin square. **(5 marks)**

Solution: Unseen.

The standard extension theorem refers to a $p \times q$ latin rectangle with entries in $\{1, \ldots, n\}$; in the present case we have $q = n - p$ so $p + q = n$ **[1]**. Let $m_L(k)$ be the number of occurrences of k in L; it is clear that $m_L(k) \geq 0$ **[1]**. The standard theorem is formulated in terms of the numbers $e_L(k) = m_L(k) + n - p - q$ **[1]**, which is just the same as $m_L(k)$ in this case **[1]**. The theorem says that L can be extended provided that $e_L(k) \geq 0$ for all k, and this is clearly satisfied. **[1]**

(10)

(a) Let $p = 4m+3$ be a prime number. Explain how to use quadratic residues modulo p to construct a block design with parameters $(4m+3, 4m+3, 2m+1, 2m+1, m)$. You should explain the key facts that need to be proved to verify that your construction works, but you do not need to prove any of them. **(7 marks)**

(b) Consider a (v, b, r, k, λ) design. Give two equations expressing r in terms of the other parameters of the design. **(2 marks)**

(c) Do there exist designs with the following parameters? Give brief reasons for your answers.

(i) $(11, 11, 5, 5, 2)$.

(ii) $(11, 11, 4, 6, 2)$.

(iii) $(11, 11, 6, 6, 2)$.

(3 marks)

Solution:

(a) **Bookwork.** Put $V = B = \mathbb{Z}/p$, so $|V| = |B| = p = 4m+3$ [1]. Let Q be the set of quadratic residues mod p [1]; it can then be shown that $|Q| = (p-1)/2 = 2m+1$ [1]. For each $j \in B$, put $C_j = j + Q \subseteq V$, so $|C_j| = |Q| = 2m+1$ [1]. The corresponding row set $R_i = \{j \mid i \in C_j\}$ is then $R_i = i - Q$, so $|R_i| = 2m+1$ as well [1]. It can be shown that when $i \neq j$ we have $|R_i \cap R_j| = m$ [1], so the sets C_j give a block design with parameters $(4m+3, 4m+3, 2n+1, 2n+1, n)$ [1].

(b) **Bookwork.** The standard relations for a block design can be written as

$$r = \frac{bk}{v}\text{[1]} = \lambda\frac{v-1}{k-1}\text{[1]}.$$

(c) For the parameters in the question we have

v	b	r	k	λ	bk/v	$\lambda(v-1)/(k-1)$
11	11	5	5	2	5	5
11	11	4	6	2	6	4
11	11	6	6	2	6	4

Case (i) is the list of parameters for the quadratic residue design mod 11 (which is $4m+3$ with $m = 2$) [1]. In cases (ii) and (iii), the relations in (b) are violated, so these cannot be the parameter list for any block design [2].